The outline

- Summary of the decompositions
- Gauge-invariant extensions
- Observability
- Accessing the OAM
- Conclusions

Reviews:

Dark spin
Quark spin

$\frac{1}{2} = S_q + L_q + S_g + L_g$

[C.L. (2013)]
[Leader, C.L. (2013)]
The decompositions in a nutshell

**Canonical**

\[ \bar{p} = \frac{\partial L}{\partial \dot{\psi}} \]

\begin{align*}
\bar{S}_q &= \frac{1}{2} \int d^3r \, \psi^{\dagger} i \bar{\Sigma} \psi \\
\bar{L}_q &= \int d^3r \, \psi^{\dagger} \bar{r} \times (-i \bar{\nabla}) \psi \\
\bar{S}_g &= \int d^3r \, \bar{E}^a \times \bar{A}^a \\
\bar{L}_g &= \int d^3r \, \bar{E}^{ai} \bar{r} \times \bar{\nabla} \bar{A}^{ai}
\end{align*}

Gauge non-invariant!

**Kinetic**

\[ \bar{\pi} = m \bar{\dot{\psi}} = \bar{p} + g \bar{A} \]

\begin{align*}
\bar{D} &= \bar{\nabla} + ig \bar{A} \\
\bar{S}_q &= \frac{1}{2} \int d^3r \, \psi^{\dagger} i \bar{\Sigma} \psi \\
\bar{L}_q &= \int d^3r \, \psi^{\dagger} \bar{r} \times (-i \bar{\nabla}) \psi \\
\bar{J}_g &= \int d^3r \, \bar{r} \times (\bar{E}^a \times \bar{B}^a)
\end{align*}

[Jaffe-Manohar (1990)]

[Ji (1997)]
The decompositions in a nutshell

The Chen et al. approach

\[ A_\mu(x) = A_{\mu}^{\text{pure}}(x) + A_{\mu}^{\text{phys}}(x) \]

Gauge transformation (assumed)

\[
A_{\mu}^{\text{pure}}(x) \mapsto U(x) \left[ A_{\mu}^{\text{pure}}(x) + \frac{i}{g} \partial_\mu \right] U^{-1}(x)
\]

\[
A_{\mu}^{\text{phys}}(x) \mapsto U(x) A_{\mu}^{\text{phys}}(x) U^{-1}(x)
\]

Pure-gauge covariant derivatives

\[
D_{\mu}^{\text{pure}} = \partial_\mu - ig A_{\mu}^{\text{pure}}(x)
\]

\[
D_{\mu}^{\text{pure}} = \partial_\mu - ig \left[ A_{\mu}^{\text{pure}}(x) \right]
\]

Field strength

\[
F_{\mu\nu}^{\text{pure}}(x) = \frac{i}{g} \left[ D_{\mu}^{\text{pure}}, D_{\nu}^{\text{pure}} \right] = 0
\]

\[
F_{\mu\nu}(x) = D_{\mu}^{\text{pure}} A_{\nu}^{\text{phys}}(x) - D_{\nu}^{\text{pure}} A_{\mu}^{\text{phys}}(x) - ig \left[ A_{\mu}^{\text{phys}}(x), A_{\nu}^{\text{phys}}(x) \right]
\]
The decompositions in a nutshell

**Canonical**

\[ p = \frac{\partial L}{\partial \dot{v}} \]

- [Jaffe-Manohar (1990)]
  \[ \tilde{S}_q = \frac{1}{2} \int d^3r \psi \dagger \tilde{\Sigma} \psi \]
  \[ \tilde{L}_q = \int d^3r \psi \dagger \vec{r} \times (-i \vec{\nabla}) \psi \]
  \[ \tilde{S}_g = \int d^3r \vec{E}^a \times \vec{A}^a \]
  \[ \tilde{L}_g = \int d^3r \vec{E}^a \dagger \vec{r} \times \vec{\nabla} A^a \]

**Kinetic**

\[ \pi = m \vec{v} = \vec{p} + \vec{g} A \]

- [Ji (1997)]
  \[ \vec{D} = \vec{\nabla} + ig \vec{A} \]
  \[ \tilde{S}_q = \frac{1}{2} \int d^3r \psi \dagger \tilde{\Sigma} \psi \]
  \[ \tilde{L}_q = \int d^3r \psi \dagger \vec{r} \times (-i \vec{D}) \psi \]
  \[ \tilde{J}_g = \int d^3r \vec{r} \times (\vec{E}^a \times \vec{B}^a) \]

- [Chen et al. (2008)]
  \[ A = A_{\text{pure}} + A_{\text{phys}} \]

- [Wakamatsu (2010)]
  \[ A = A_{\text{pure}} + A_{\text{phys}} \]
  \[ \tilde{S}_q = \frac{1}{2} \int d^3r \psi \dagger \tilde{\Sigma} \psi \]
  \[ \tilde{L}_q = \int d^3r \psi \dagger \vec{r} \times (-i \vec{D}_{\text{pure}}) \psi \]
  \[ \tilde{S}_g = \int d^3r \vec{E}^a \times \vec{A}_{\text{phys}}^a \]
  \[ \tilde{L}_g = \int d^3r \vec{E}^a \dagger \vec{r} \times \vec{D}_{\text{pure}} A^a \]

\[ \int d^3r \vec{r} \times [(\vec{A}_{\text{phys}}^a \times \vec{D}_{\text{pure}}) \times \vec{E}^a] - \int d^3r \vec{E}^a \times \vec{A}_{\text{phys}}^a \]

**Gauge non-invariant!**

**Gauge-invariant extension (GIE)**
The decompositions in a nutshell

### Canonical

\[ \vec{p} = \frac{\partial L}{\partial \dot{\vec{v}}} \]

- \( S_q \)
- \( S_g \)
- \( L_g \)
- \( L_q \)

\[ \vec{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \Sigma \psi \]
\[ \vec{L}_q = \int d^3 r \psi^\dagger \vec{r} \times (-i \vec{\nabla}) \psi \]
\[ \vec{S}_g = \int d^3 r \vec{E}^a \times \vec{A}^a \]
\[ \vec{L}_g = \int d^3 r E^{ai} \vec{r} \times \vec{\nabla} A^{ai} \]

Gauge non-invariant!

### Kinetic

\[ \vec{\pi} = m \vec{\dot{v}} = \vec{p} + g \vec{A} \]

\[ \vec{D} = \vec{\nabla} + ig \vec{A} \]

- \( J_g \)
- \( J_q \)
- \( S_q \)
- \( S_g \)
- \( L_g \)
- \( L_q \)

\[ \vec{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \Sigma \psi \]
\[ \vec{L}_q = \int d^3 r \psi^\dagger \vec{r} \times (-i \vec{D}) \psi \]
\[ \vec{J}_g = \int d^3 r \vec{r} \times (\vec{E}^a \times \vec{B}^a) \]

Gauge-invariant extension (GIE)

\[ \vec{L}_{pot} = \int d^3 r \rho^a \vec{r} \times \vec{A}_{phys}^a \]
\[ \rho^a = g \psi^\dagger t^a \psi = (\vec{D} \cdot \vec{E})^a \]

- \( A = A_{pure} + A_{phys} \)

### Explanations

- **Jaffe-Manohar (1990):**
  \[ \vec{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \Sigma \psi \]
  \[ \vec{L}_q = \int d^3 r \psi^\dagger \vec{r} \times (-i \vec{D}_{pure}) \psi \]
  \[ \vec{S}_g = \int d^3 r \vec{E}^a \times \vec{A}_{phys}^a \]
  \[ \vec{L}_g = \int d^3 r E^{ai} \vec{r} \times \vec{D}_{pure} A^{ai} \]

- **Chen et al. (2008):**
  \[ \vec{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \Sigma \psi \]
  \[ \vec{L}_q = \int d^3 r \psi^\dagger \vec{r} \times (-i \vec{D}_{pure}) \psi \]
  \[ \vec{S}_g = \int d^3 r \vec{E}^a \times \vec{A}_{phys}^a \]
  \[ \vec{L}_g = \int d^3 r E^{ai} \vec{r} \times \vec{D}_{pure} A^{ai} \]

- **Ji (1997):**
  \[ \vec{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \Sigma \psi \]
  \[ \vec{L}_q = \int d^3 r \psi^\dagger \vec{r} \times (-i \vec{D}) \psi \]
  \[ \vec{J}_g = \int d^3 r \vec{r} \times (\vec{E}^a \times \vec{B}^a) \]

- **Gauge-invariant extension (GIE):**
  \[ \int d^3 r \vec{r} \times [(\vec{A}_{phys}^a \times \vec{D}_{pure}) \times \vec{E}^a] \]
  \[ - \int d^3 r \vec{E}^a \times \vec{A}_{phys}^a \]

- **Wakamatsu (2010):**
  \[ \vec{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \Sigma \psi \]
  \[ \vec{L}_q = \int d^3 r \psi^\dagger \vec{r} \times (-i \vec{D}_{pure}) \psi \]
  \[ \vec{S}_g = \int d^3 r \vec{E}^a \times \vec{A}_{phys}^a \]
  \[ \vec{L}_g = \int d^3 r E^{ai} \vec{r} \times \vec{D}_{pure} A^{ai} \]

- **C.L. (2013):**
  \[ \vec{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \Sigma \psi \]
  \[ \vec{L}_q = \int d^3 r \psi^\dagger \vec{r} \times (-i \vec{D}) \psi \]
  \[ \vec{J}_g = \int d^3 r \vec{r} \times (\vec{E}^a \times \vec{B}^a) \]

- **[C.L. (2013)]:**
  \[ \int d^3 r \vec{r} \times [(\vec{A}_{phys}^a \times \vec{D}_{pure}) \times \vec{E}^a] \]
  \[ - \int d^3 r \vec{E}^a \times \vec{A}_{phys}^a \]
The Stueckelberg symmetry

\[ A = A_{\text{pure}} + A_{\text{phys}} = \overline{A}_{\text{pure}} + A_{\text{phys}} - C \]

Ambiguous!
Infinetly many possibilities!

Coulomb GIE

\[ \mathcal{D}_{\text{pure}} \cdot \overline{A}_{\text{phys}} = 0 \]

Light-front GIE

\[ A^+_{\text{phys}} = 0 \]

[Chen et al. (2008)]

[Hatta (2011)]

[Wakamatsu (2010)]

[Stoilov (2010)]

[C.L. (2013)]

[Chen et al. (2008)]

[Wakamatsu (2010)]
The gauge-invariant extension (GIE)

Gauge-variant operator

GIE1

GIE2

Gauge

« Natural » gauges

Lorentz-invariant extensions

Rest

Center-of-mass

Infinite momentum

« Natural » frames

\[ p^2 = m_0^2 \]

\[ s = E_{CM}^2 \]

\[ x = k_{IMF}^z / p_{IMF}^z \]
The geometrical interpretation

Parallel transport

\[ \mathcal{W}(x + dx, x) = 1 + igA_\mu(x)dx^\mu \]

\[ \mathcal{W}_C(x + dx, x_0)\mathcal{W}_C(x_0, x) = 1 + igA_{\mu}^{\text{pure}}(x)dx^\mu \]

\[ A_{\mu}^{\text{pure}}(x) = \frac{i}{g} \mathcal{W}_C(x, x_0) \frac{\partial}{\partial x^\mu} \mathcal{W}_C(x_0, x) \]

\[ A_{\mu}^{\text{phys}}(x) = -\int_{x_0}^{x} \mathcal{W}_C(x, s)F_{\alpha\beta}(s)\mathcal{W}_C(s, x) \frac{\partial s^\alpha}{\partial x^\mu} ds^\beta \]

Path dependence \rightarrow Stueckelberg dependence

[Non-local!]
The semantic ambiguity

Quid?

« physical » ↔ « measurable »

« gauge invariant »

Observables

*E.g.* cross-sections

Expansion scheme

*E.g.* collinear factorization

Path

Stueckelberg Background
dependent but fixed by the process

Measurable, physical, gauge invariant and local

Quasi-observables

*E.g.* parton distributions

« Measurable », « physical », gauge invariant but non-local

Light-front gauge links
The observability

**Observable**  **Quasi-observable**  **Not observable**

### Canonical

- **[Jaffe-Manohar (1990)]**
  \[
  \mathcal{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \bar{\Sigma} \psi \\
  \mathcal{L}_q = \int d^3 r \psi^\dagger \bar{r} \times (-i \bar{\nabla}) \psi \\
  \mathcal{S}_g = \int d^3 r \bar{E}^a \times \bar{A}^a \\
  \mathcal{L}_g = \int d^3 r \bar{E}^{ai} \bar{r} \times \bar{\nabla} \bar{A}^{ai}
  \]

- **[Chen et al. (2008)]**
  \[
  \mathcal{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \bar{\Sigma} \psi \\
  \mathcal{L}_q = \int d^3 r \psi^\dagger \bar{r} \times (-i \bar{\nabla}_{\text{pure}}) \psi \\
  \mathcal{S}_g = \int d^3 r \bar{E}^a \times \bar{A}_{\text{phys}}^a \\
  \mathcal{L}_g = \int d^3 r \bar{E}^{ai} \bar{r} \times \bar{D}_{\text{pure}} \bar{A}_{\text{phys}}^{ai}
  \]

### Kinetic

- **[Ji (1997)]**
  \[
  \mathcal{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \bar{\Sigma} \psi \\
  \mathcal{L}_q = \int d^3 r \psi^\dagger \bar{r} \times (-i \bar{D}) \psi \\
  \mathcal{J}_g = \int d^3 r \bar{r} \times (\bar{E}^a \times \bar{B}^a)
  \]

- **[Wakamatsu (2010)]**
  \[
  \mathcal{S}_q = \frac{1}{2} \int d^3 r \psi^\dagger \bar{\Sigma} \psi \\
  \mathcal{L}_q = \int d^3 r \psi^\dagger \bar{r} \times (-i \bar{D}) \psi \\
  \mathcal{S}_g = \int d^3 r \bar{E}^a \times \bar{A}_{\text{phys}}^a \\
  \mathcal{L}_g = \int d^3 r \bar{r} \times (\bar{E}^a \times \bar{B}^a)
  \int d^3 r \bar{r} \times [(\bar{A}_{\text{phys}}^a \times \bar{D}_{\text{pure}}) \times \bar{E}^a] \\
  - \int d^3 r \bar{E}^a \times \bar{A}_{\text{phys}}^a
  \]
The gluon spin

\[ \Delta g = \int_0^1 dx \, \Delta g(x) \]

\[ = \int_0^1 dx \, \frac{i}{xP^+} \int \frac{dz^-}{2\pi} \, e^{ixP^+z^-} \langle P, \Lambda | 2 \text{Tr}[F^{+\alpha}(0)W_{0,z^-}] \tilde{F}^{+\alpha}(z^-)W_{z^-,0}] | P, \Lambda \rangle \]

\[ = \frac{\epsilon^{+\alpha\beta}}{2P^+} \langle P, \Lambda | 2 \text{Tr}[F^{+\alpha}(0) \int dz^- \frac{1}{2} \epsilon(z^-) W_{0,z^-} F^{+\beta}(z^-)W_{z^-,0}] | P, \Lambda \rangle \]

« Measurable », gauge invariant but non-local

\[ \text{Light-front gauge} \quad A^+ = 0 \]

\[ \text{Light-front GIE} \quad A^+_{\text{phys}} = 0 \]

\[ \text{[Jaffe-Manohar (1990)]} \]

\[ = \frac{\epsilon^{+\alpha\beta}}{2P^+} \langle P, \Lambda | 2 \text{Tr}[F^{+\alpha}(0)A^{\beta}(0)] | P, \Lambda \rangle \]

\[ = \frac{1}{2P^+} \langle P, \Lambda | \frac{1}{2} \epsilon^{+\alpha\beta} M^{+\alpha\beta,\text{JY}}(0) | P, \Lambda \rangle \]

\[ = \frac{\langle P, \Lambda | S_{g,\text{spin}}^z | P, \Lambda \rangle}{\langle P, \Lambda | P, \Lambda \rangle} \]

Local fixed-gauge interpretation

\[ \text{[Hatta (2011)]} \]

\[ = \frac{\epsilon^{+\alpha\beta}}{2P^+} \langle P, \Lambda | 2 \text{Tr}[F^{+\alpha}(0)A^{\beta,\text{Hatta}}_{\text{phys}}(0)] | P, \Lambda \rangle \]

\[ = \frac{1}{2P^+} \langle P, \Lambda | \frac{1}{2} \epsilon^{+\alpha\beta} M^{+\alpha\beta,\text{Hatta}}(0) | P, \Lambda \rangle \]

\[ = \frac{\langle P, \Lambda | S_{g,\text{Hatta}}^z | P, \Lambda \rangle}{\langle P, \Lambda | P, \Lambda \rangle} \]

Non-local gauge-invariant interpretation
The kinetic and canonical OAM

**Kinetic OAM (Ji)**

\[
L_z = \frac{1}{2} \int \frac{d x \cdot d y}{J_z} \left[ H(x, 0, 0) + E(x, 0, 0) \right] - \frac{1}{2} \int \frac{d x \cdot d y}{S_z} \left[ H(x, 0, 0) \right]
\]

Quark *naive* canonical OAM (Jaffe-Manohar)

\[
L_z = - \int \frac{d x \cdot d y}{k_\perp} \left[ H(x, 0, 0) + E(x, 0, 0) + \tilde{E}_2(x, 0, 0) \right]
\]

*Pure twist-3*

Quark *naive* canonical OAM (Jaffe-Manohar)

\[
L_z = - \int \frac{d x \cdot d y}{k_\perp} \left[ H(x, 0, 0) + E(x, 0, 0) + \tilde{E}_2(x, 0, 0) \right]
\]

Canonical OAM (Jaffe-Manohar)

\[
\ell_z = - \int \frac{d x \cdot d y}{k_\perp} \left[ h_4(x, k_\perp) \right]
\]

<table>
<thead>
<tr>
<th>Model</th>
<th>LCCQM</th>
<th>(\chi)QSM</th>
</tr>
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<td></td>
<td>(u)</td>
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<td>(\ell_q^q)</td>
<td>0.131</td>
<td>-0.005</td>
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<td>(L_q^q)</td>
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<td>0.055</td>
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<tr>
<td>(L_q^q)</td>
<td>0.169</td>
<td>-0.042</td>
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</tbody>
</table>

\[\ell_z = L_z \quad \text{but} \quad \ell_q^q \neq L_q^q\]

![No gluons and not QCD EOM!](image_url)
The phase-space picture

Complete parametrizations:
- Quarks [Meissner, Metz, Schlegel (2009)]
- Quarks & gluons [C.L., Pasquini (2013)]

GTMDs
TMDs
FFs
PDFs
Charges
GPDs

2+3D → 0+3D
0+1D
2+0D
0+1D
2+1D

k^+ = xP^+

[f d^2 b_⊥, f d^2 k_⊥, f d^x]
Average transverse quark momentum in a longitudinally polarized nucleon

\[ \langle \vec{k}_\perp \rangle (b_\perp) = \int dx \, d^2 k_\perp \, \vec{k}_\perp \, \rho_+^{[\gamma^+]} (x, \vec{k}_\perp, \vec{b}_\perp) \]

\[ \langle \bar{k}_\perp^n \rangle \]

\[ \langle \bar{k}_\perp^d \rangle \]

\[ F_{14} \]

« Vorticity »
The conclusions

• Kinetic and canonical decompositions are physically inequivalent and are both interesting

• Measurability requires gauge invariance but not necessarily local expressions

• All the canonical and kinetic contributions are measurable (twist-3 GPDs, GTMDs?) and computable on a lattice

Reviews:
[C.L. (2013)]
[Leader, C.L. (2013)]
Backup slides
The path dependence

Orbital angular momentum

\[ \ell_z = \frac{\langle p, + | \hat{L}_z | p, + \rangle}{\langle p, + | p, + \rangle} \]

\[ \hat{L}_z = \int d^4r \, \delta(r^+) \, \bar{\psi}(r) \gamma^+ \left( \vec{r}_\perp \times i \vec{D}_\perp^{\text{pure}} \right)_z \psi(r) \]

\[ = \int d^4r \, \delta(r^+) \, \bar{\psi}_D(r) \gamma^+ \left( \vec{r}_\perp \times (-i) \vec{\nabla}_\perp \right)_z \psi_D(r) \]

\[ D^\text{pure}_\mu(y) = \partial_\mu - ig A^\text{pure}_\mu(y) \]

\[ = \partial_\mu - ig \left[ \frac{i}{g} \mathcal{W}_{[y,y_0]} \partial_\mu \mathcal{W}_{[y_0,y]} \right] \]

\[ = \mathcal{W}_{[y,y_0]} \psi_{D}(y) \rightarrow \text{Reference point} \]

Canonical

[Jaffe, Manohar (1990)]

\[ D^\text{pure}_\mu A^+ = 0 \quad \partial_\mu \]

\[ \psi_D(r) A^+ = 0 \quad \psi(r) \]

Kinetic

[Ji (1997)]

\[ D^\text{pure}_\mu(r) = D_\mu(r) \]

\[ D^\text{pure}_\mu(y) \neq D_\mu(y) \quad y \neq r \]

\[ \mathcal{W}_{\text{straight}}^{[r,y]} \]

\[ \ell^\text{DY} = \ell^\text{SIDIS} \]
The quark orbital angular momentum

\[ W_{A'A'}^{[\Gamma]}(x, \xi, \vec{k}_\perp, \vec{\Delta}_\perp) = \frac{1}{2} \langle p', A' | \hat{W}^{[\Gamma]}(0, x P^+, \vec{k}_\perp) | p, A \rangle \]

Wigner distribution

\[ \rho_{A'A'}^{[\Gamma]}(x, \vec{k}_\perp, \vec{b}_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp} W_{A'A'}^{[\Gamma]}(x, 0, \vec{k}_\perp, \vec{\Delta}_\perp) \]

Orbital angular momentum

\[ \ell_z = \int dx \, d^2 k_\perp \, d^2 b_\perp \left( \vec{b}_\perp \times \vec{k}_\perp \right)_z \rho_{++}^{[\gamma^+]}(x, \vec{k}_\perp, \vec{b}_\perp) \]

Unpolarized quark density

\[ = \int dx \, d^2 k_\perp \left( \vec{k}_\perp \times i \vec{\nabla}_{\Delta_\perp} \right)_z W_{++}^{[\gamma^+]}(x, 0, \vec{k}_\perp, \vec{\Delta}_\perp) \bigg|_{\vec{\Delta}_\perp=\vec{0}_\perp} \]

\[ = - \int dx \, d^2 k_\perp \frac{\vec{k}_\perp^2}{M^2} F_{14}(x, 0, \vec{k}_\perp, \vec{0}_\perp) \]

Parametrization

\[ W_{A'A'}^{[\gamma^+]} = \frac{1}{2M} \bar{u}(p', A') \left[ F_{11} + \frac{i \sigma_{\perp}^k}{P^+} F_{12} + \frac{i \sigma_{\perp}^\Delta}{P^+} F_{13} + \frac{i \sigma_{\perp}^k \Delta}{M^2} F_{14} \right] u(p, A) \]

[Meißner, Metz, Schlegel (2009)]
The emerging picture

Longitudinal

\[ g_{1L} \leftrightarrow \tilde{H} \]
\[ \ell_z \leftrightarrow F_{14} \]
\[ C_z \leftrightarrow G_{11} \]

[C.L., Pasquini (2011)]

Transverse

\[ h_1 \leftrightarrow H_T \]
\[ f_{1T}^{\perp} \leftrightarrow E \]
\[ h_1^{\perp} \leftrightarrow 2\tilde{H}_T + E_T \]

[Barone et al. (2008)]
The gauge symmetry

Quantum electrodynamics

\[ \psi(x) \quad \hat{\psi}_{\text{phys}}(x) = U_{\text{pure}}^{-1}(x)\psi(x) \]

**Passive**

\[ \psi(x) \leftrightarrow U(x)\psi(x) \]

\[ U_{\text{pure}}(x) \leftrightarrow U(x)U_{\text{pure}}(x) \]

\[ \hat{\psi}_{\text{phys}}(x) \leftrightarrow \hat{\psi}_{\text{phys}}(x) \]

**Active**

\[ \psi(x) \leftrightarrow U(x)\psi(x) \]

\[ U_{\text{pure}}(x) \leftrightarrow U_{\text{pure}}(x) \]

\[ \hat{\psi}_{\text{phys}}(x) \leftrightarrow U(x)\hat{\psi}_{\text{phys}}(x) \]

**Active \times (Passive)^{-1}**

\[ \psi(x) \leftrightarrow \psi(x) \]

\[ U_{\text{pure}}(x) \leftrightarrow U_{\text{pure}}(x)U^{-1}(x) \]

\[ \hat{\psi}_{\text{phys}}(x) \leftrightarrow U(x)\hat{\psi}_{\text{phys}}(x) \]

[C.L. (2013)]

« Physical »

« Background »
The phase-space distribution

Wigner distribution

\[ \rho(r, k) = \int \frac{dz}{2\pi} e^{-ikz} \psi^*(r - \frac{z}{2})\psi(r + \frac{z}{2}) \]
\[ = \int \frac{d\Delta}{(2\pi)^2} e^{-i\Delta r} \varphi^*(k + \frac{\Delta}{2})\varphi(k - \frac{\Delta}{2}) \]

Probabilistic interpretation

\[ \int dk \rho(r, k) = |\psi(r)|^2 \]
\[ \int dr \rho(r, k) = |\varphi(k)|^2 \]

Expectation value

\[ \langle \hat{O} \rangle = \int dr \psi^*(r)O(r, -i\partial_r)\psi(r) \]
\[ = \int \frac{dk}{2\pi} \varphi^*(k)O(i\partial_k, k)\varphi(k) \]
\[ = \int dr dk O(r, k)\rho(r, k) \]

Galilei covariant

- Either non-relativistic
- Or restricted to transverse position

[H. Wigner (1932)]
[M. Moyal (1949)]
### Parametrization @ twist-2 and $\xi=0$

#### Quark polarization

<table>
<thead>
<tr>
<th></th>
<th>$U$</th>
<th>$T_x$</th>
<th>$T_y$</th>
<th>$L$</th>
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<tbody>
<tr>
<td>$U$</td>
<td>$F_{11}$</td>
<td>$\frac{i}{M} (k_y H_{11} + \Delta_y H_{12})$</td>
<td>$-\frac{i}{M} (k_x H_{11} + \Delta_x H_{12})$</td>
<td>$\frac{i(\Delta_{\perp} \times k_{\perp}) \cdot G_{11}}{M^2}$</td>
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<tr>
<td>$T_x$</td>
<td>$\frac{i}{M} \left( k_y F_{11} + \Delta_y (F_{13} - \frac{1}{2} F_{11}) \right)$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
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<tr>
<td>$T_y$</td>
<td>$-\frac{i}{M} \left( k_x F_{11} + \Delta_x (F_{13} - \frac{1}{2} F_{11}) \right)$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
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<tr>
<td>$L$</td>
<td>$\frac{1}{M} (k_x H_{17} + \Delta_x H_{18})$</td>
<td>$\frac{1}{M} (k_y H_{17} + \Delta_y H_{18})$</td>
<td>$G_{14}$</td>
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### Nucleon polarization

$\Delta_\perp = \tilde{\Theta}_\perp$

$f \, d^2 k_\perp$

#### GTMDs

<table>
<thead>
<tr>
<th></th>
<th>$U$</th>
<th>$T_x$</th>
<th>$T_y$</th>
<th>$L$</th>
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<tbody>
<tr>
<td>$U$</td>
<td>$f_1$</td>
<td>$\frac{k_y}{M} h_{11}$</td>
<td>$-\frac{k_x}{M} h_{11}$</td>
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<td>$T_x$</td>
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<td>$h_1 + \frac{k_x^2}{2M^2} h_{1T}$</td>
<td>$\frac{k_x k_y}{M^2} h_{1T}$</td>
<td>$\frac{k_x}{M} g_{1T}$</td>
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<tr>
<td>$T_y$</td>
<td>$-\frac{k_x}{M} f_{1T}$</td>
<td>$\frac{k_x}{M^2} h_{1T}$</td>
<td>$h_1 - \frac{k_x^2}{2M^2} h_{1T}$</td>
<td>$\frac{k_y}{M} g_{1T}$</td>
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<tr>
<td>$L$</td>
<td>$\frac{k_x}{M} h_{1L}$</td>
<td>$\frac{k_y}{M} h_{1L}$</td>
<td>$g_{1L}$</td>
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#### TMDs

#### GPDs

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<tbody>
<tr>
<td>$U$</td>
<td>$H$</td>
<td>$\frac{\Delta_y}{2M} (2\tilde{H}_T + E_T)$</td>
<td>$-\frac{\Delta_x}{2M} (2\tilde{H}_T + E_T)$</td>
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<td>$T_x$</td>
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<td>$H_T - \frac{\Delta^2_y}{4M^2} \tilde{H}_T$</td>
<td>$-\frac{\Delta_x \Delta_y}{2M^4} \tilde{H}_T$</td>
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<td>$T_y$</td>
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<td>$-\frac{\Delta_x \Delta_y}{2M^4} \tilde{H}_T$</td>
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<td>$L$</td>
<td>$\tilde{H}$</td>
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#### Monopole

#### Dipole

#### Quadrupole
OAM and origin dependence

Naive
\[ \mathcal{L}_{iz} = \vec{r}_{i\perp} \times \vec{k}_{i\perp} \]

- Depends on proton position

Momentum conservation
\[ \sum_{i=1}^{N} \vec{k}_{i\perp} = \vec{0}_{\perp} \]

Physical interpretation?

Relative
\[ \ell_{iz}^{\text{rel}} = \vec{\rho}_{i\perp} \times \vec{k}_{i\perp} \]

Intrinsic
\[ \ell_{iz}^{\text{int}} = \vec{b}_{i\perp} \times \vec{k}_{i\perp} \]

Transverse center of momentum
\[ \vec{R}_{\perp} = \sum_{i=1}^{N} x_i \vec{r}_{i\perp} \]

Equivalence
\[ \mathcal{L}_z = \ell_z^{\text{rel}} = \ell_z^{\text{int}} \]

- Intrinsic
\[ \sum_{i=1}^{N} \vec{b}_{i\perp} \times \vec{k}_{i\perp} = \sum_{i=1}^{N} \left( \vec{r}_{i\perp} - \vec{R}_{\perp} \right) \times \vec{k}_{i\perp} \]

- Naive
\[ \sum_{i=1}^{N} \vec{b}_{i\perp} \times \vec{k}_{i\perp} = \sum_{i=1}^{N} \vec{r}_{i\perp} \times \vec{k}_{i\perp} - \vec{R}_{\perp} \times \sum_{i=1}^{N-1} \vec{k}_{i\perp} \]

- Relative
\[ \sum_{i=1}^{N-1} \vec{\rho}_{i\perp} \times \vec{k}_{i\perp} \]
Overlap representation

Fock expansion of the proton state

\[ |p\rangle = \Psi_{qq} |qq\rangle + \Psi_{qqg} |qqg\rangle + \Psi_{qqqg} |qqq\rangle + \Psi_{qqqq\bar{q}} |qqqq\bar{q}\rangle + \cdots \]

Fock states

Simultaneous eigenstates of

\[ P^+ = \sum_{i=1}^{N} k_i^+ \]

\[ \vec{0}_\perp = \vec{P}_\perp = \sum_{i=1}^{N} \vec{k}_\perp \]

Momentum

Light-front helicity
Overlap representation

Fock-state contributions

Kinetic OAM

\[ L_z^{N\beta,q} = \frac{1}{2} \int [dx]_N [d^2k_\perp]_N \sum_{i=1}^{N} \delta_{qq_i} \left\{ (x_i - \lambda_i)|\Psi_{N\beta}^\uparrow|^2 + Mx_i \sum_{n=1}^{N} (\delta_{ni} - x_n) \left[ \Psi_{N\beta}^\uparrow \frac{\partial}{\partial k_n} \Psi_{N\beta} \right] \right\} \]

Naive canonical OAM

\[ \mathcal{L}_z^{N\beta,q} = -\frac{i}{2} \int [dx]_N [d^2k_\perp]_N \sum_{i=1}^{N} \delta_{qq_i} \left[ \Psi_{N\beta}^\uparrow \left( \vec{k}_i \times \vec{\nabla}_{k_i} \right) z \Psi_{N\beta} \right] \]

Canonical OAM

\[ \ell_z^{N\beta,q} = -\frac{i}{2} \int [dx]_N [d^2k_\perp]_N \sum_{i=1}^{N} \delta_{qq_i} \sum_{n=1}^{N} (\delta_{ni} - x_n) \left[ \Psi_{N\beta}^\uparrow \left( \vec{k}_i \times \vec{\nabla}_{k_n} \right) z \Psi_{N\beta} \right] \]