MCG International Conference 2017

MCG 10: 10TH INTERNATIONAL CONFERENCE

# Mathematical Creativity and Giftedness

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# PROCEEDINGS

**Editor**

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# The 10th Mathematical Creativity and Giftedness International Conference

# PROCEEDINGS

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**WORKSHOPS**

# WELCOME NOTE

On behalf of the International and Local Organizing Committee, it is my pleasure to welcome you to the 10th International Mathematical Creativity and Giftedness Conference (MCG10).

The MCG Conference provides the forum to its participants to present and discuss research ideas and findings pertinent to mathematical creativity and giftedness. The main topics of the conference focus on mathematical creativity for all students, from all backgrounds and of all ages, creativity inside and outside the classroom, aptitude and achievement, giftedness, talent and promise, and mathematics competitions. The conference serves as a context where research experience interweaves with theory and practice. Mathematics educators, researchers, mathematicians and teachers as well as professionals with interdisciplinary interests have the opportunity to interact and share knowledge pertinent to mathematical creativity and giftedness.

As the conference chair of MCG 10, I would like to express my sincere thanks to the President of MCG Professor Roza Leikin and to the members of the International Program Committee Professor Marianna Nolte, Professor Linda Sheffield and Professor Peter Taylor. I would also like to thank the International Scientific Committee and the Local Organizing Committee.

Finally, I would like to express my appreciation to the co-organizer the Cyprus Mathematical Society. In addition to this, I would like to thank the sponsors of the MCG 10, the Cyprus Tourism Organization, the Austrian Airlines and the University of Cyprus.

Associate Professor Demetra Pitta-Pantazi
MCG 10 Organizing Chair
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# PLENARY LECTURES

# DEVELOPING MATHEMATICAL CREATIVITY AND EXPERTISE IN STUDENTS AND TEACHERS: FOCUSING ON MULTIPLE SOLUTION AND INVESTIGATION TASKS

Roza Leikin
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**Abstract.**(1) *I consider developing mathematical expertise and mathematical creativity in each student as two equivalent purposes of school mathematics education. Multiple Solution Tasks and Investigation Tasks are exemplified as main tools for the identification and development of students' mathematical expertise and creativity. While development of mathematical expertise is not necessarily applies advancement of mathematical creativity, creativity-directed learning advances problem-solving expertise. I argue that development of creativity in students entails the creation of a "creative" learning environment and thus requires expertise and creativity in mathematics teachers.*

## Introduction

In his arguments regarding the importance of creativity for child development, Vygotsky (1982/1930, 1984/1931) maintains that creativity (or 'imagination' in Vygotsky's words) is the central mechanism in the development of children's knowledge, since it allows them to construct connections between their existing knowledge and new pieces of information. In contrast, creative processes presume the discovery of new constructs, properties and regularities to expand existing knowledge to the new territory, on the basis of one's existing knowledge. This observation leads directly to a *knowledge-creativity paradox*: creativity is a necessary condition for knowledge construction, whereas knowledge is a necessary condition for creative processing. This intriguing phenomenon leads to a question that is surprisingly overlooked in mathematics education research: Does higher creativity in individuals lead to more advanced mathematical knowledge/expertise, or is mathematical creativity a function of the knowledge/expertise development? Consequently mutual relationships between mathematical creativity and mathematical expertise has been the focus of my research during the past decade.

In his seminal lecture before the French Psychological Society, Poincare (1908) provided analysis of mathematical creation in professional mathematicians who have "the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance. This is a true esthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility" (ibid. p. 92).This esthetic feeling, feeling of beauty and harmony should be developed in students by means of creative activities in mathematics classroom.

(1) This paper is a part of the earlier publication: Leikin, R. (2016). Interplay between creativity and expertise in teaching and learning of mathematics. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education (*Vol. 1, pp. 19–34). Szeged, Hungary: PME.

## BACKGROUND

### Mathematical potential and expertise

Mathematics education is aimed at providing equal learning opportunities to all students (NCTM, 1989) which enable the realization of learners' mathematical potential to the maximal extent. Milgram & Hong (2009) stressed the danger of "talent loss" which is "the failure of individuals to realize the potential for the extraordinary achievement in a specific domain that they demonstrated in their early years" (p. 149). Clearly, students differ in their mathematical potential and, thus, according to the equity principle of mathematics education, "reasonable and appropriate accommodations (should) be made to promote access and attainment for all students" (NCTM, 2000, p. 12). Realization of mathematical potential is a function of mathematical abilities, motivation, personality and the learning opportunities open to the students at different stages of their development (Leikin, 2009).

Krutetskii (1968/1976) designed 26 unique series of problems containing 79 tests with problems of varying difficulty, requiring mental processing that characterizes the work of professional mathematicians. Comparison of the problem solving performance by "capable, average and incapable" school students led to the understanding of special characteristics of several key components essential to high mathematical ability: • ability to memorize mathematical objects, schemes, principles, and relationships; • ability to grasp formal structures; • ability to think logically in spatial, numeric, and symbolic relationships; • ability to switch from direct to reverse trains of thought; • ability to generalize rapidly and broadly; • ability to be flexible with mental processes; • ability to appreciate clarity, simplicity, and rationality.

The analysis of individual cases of mathematically gifted students ("whose mathematical abilities began to appear, as a rule, at early ages") illustrates that the concept of mathematical cast of mind as connected to mathematical insight. The following problem was presented to the students: *A father is 35 years old and his son is 2. In how many years will the father be 4 times as old as his son?*

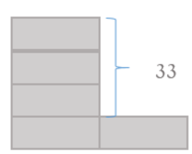
| Gilya solved it as follows | Volodya solved it using diagram |
|---|---|
| *He is 33 years older, that means one part will be 11, and the son still needs 9 more years.* |  |

Figure 1

Usually school mathematics tests examine students' ability to cope with learning-based problems only and rarely examine creativity or mathematical insight.

## Creative talent in mathematics

Milgram & Hong (2009) argue that analysis of the activities of a number of eminent mathematicians demonstrated that inventions and accomplishments in mathematics require creative talent rather than traditional academic ability. Milgram and Hong suggested a comprehensive model of talent development that introduced a distinction between two types of talent: *expert talent* and *creative talent*. Another view on the connection between mathematical giftedness and creativity was drawn by (Usiskin, 2000) in the eight-tiered (from 0 to 7) hierarchy of mathematical gift ( Sriraman, 2005) as a function of mathematical creativity.

An expert has been described as having more robust mental imagery, more numerous images, the ability to switch efficiently and effectively between different images, the ability to focus attention on appropriate features of problems, and having more cognizance of their own thought process and of how others may think (Carlson & Bloom, 2005; Lester, 1994;Ericson, 1996). In contrast to experts, a novice's system of representations of a mathematical concept may be deficient in number and in connections to form an adequate network of knowledge (Lester 1994). Experts differ from novices in the problem-solving strategies they employ (Schoenfeld, 1992) and in their ability to categorize problems according to solution principles and choose the most efficient ways of solutions for a particular type of problem (Sweller, Mawer, and Ward, 1983).

Researchers make a distinction between algorithmic, strategic and creative problem solving, while creative problem solving is associated with mental flexibility (Silver 1997; Star and Newton 2009) and with mathematical insight (Krutetskii, 1976; Ervynck 1991; Leikin, 2013). Insight-based problems are defined as problems that have a relatively simple solution which is difficult to discover until solution-relevant features are recognized (Weisberg 2015).

## Multiple Solution Tasks and Investigation Tasks as tools for the development and identification creativity

Silver (1997) suggested developing creativity through problem solving as follows: *Fluency* is developed by generating multiple mathematical ideas, multiple answers to a mathematical problem (when such exist), and exploring mathematical situations. *Flexibility* is advanced by generating new mathematical solutions when at least one has already been produced. *Originality* is advanced by exploring many solutions to a mathematical problem and generating a new one. Below I present several examples.

## Examples of creativity-directed mathematical tasks

A *multiple-solution task* (MST) is one in which learners are explicitly required to solve a mathematical problem using multiple solution strategies. The distinctions between the solution strategies can be based, for example, on (a) use of different representations of a mathematical concept; (b) use of different properties (definitions or theorems) of a mathematical concept; (c) use of mathematical tools from different branches of mathematics; and (d) use of tools from different fields (see examples (Leikin, 2007) in Figure 2). In this context *solution spaces of MSTs* (cf. example spaces in Watson & Mason, 2005) are used for the analysis of problem-solving performance associated with MSTs. Expert solution spaces are the most complete collection of solutions of a problem at a given moment. Individual solution spaces include solutions that a participant may produce on the spot or after several attempts. Usually individual solution spaces are subsets of expert solution spaces; however, sometimes they can broaden expert solution spaces. Collective solution spaces characterize solutions produced by a group of solvers. They are usually broader than personal solution spaces and within a particular group and are one of the main sources for the development of individual spaces. All the solution spaces can include conventional (i.e. learning-based) and unconventional (not learning-based that usually require insight) solutions (see Figure 2).

**Tasks:** Solve the problem in at least 3 different ways

| Learning-based solutions | Insight-based solutions |
|---|---|
| P1. Calculation: $2\frac{1}{4} \times 1.75$ | |
| 1.1 The distributive law:<br>-in decimal numbers;<br>-in common fraction;<br>1.2 Vertical multiplication;<br>1.3 Net multiplication | 1.4 Reduced multiplication<br>$2\frac{1}{4} \times 1.75 = \left(2 + \frac{1}{4}\right)\left(2 - \frac{1}{4}\right)$ |
| P2. **Jam problem:** Mali produces strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. One time she distributed 80 liters of jam equally among the jars. She decided to save 4 jars and to distribute jam from these jars equally among the other jars. She realized that she had added exactly 1/4 of the previous amount to each of the jars. How many jars did she prepare at the start? | |
| 2.1 System of equations with two variables;<br>2.2 Equation with one variable;<br>2.3 Fractions/ Percentages. | 2.4 Diagram based<br>2.5 Elegant equations<br>2.6 Logical:<br>*4 jars include 1/4 of all the jam. Thus there were 20 jars.* |
| P3 $\begin{cases} 4x + 5y = 18 \\ 5x + 4y = 18 \end{cases}$ | |
| 3.1 Algebraic combination.<br>3.2 Substitution.<br>3.3 Subtraction/ addition of equations<br>3.4 Graphing.<br>3.5 Solutions with determinants. | 3.6 Symmetry based consideration<br>*The exchange of variables does not change the system which has only one solution:* $x = y = 2$<br>(Polya, 1973) |
| Half-Time -- Half-Way problem: Dan and Tal walk from the train station to the hotel. They start out at the same time. Dan walks half the time at speed $v_1$ and half the time at speed $v_2$. Tal walks half way at speed $v_1$ and half way at speed $v_2$. Who gets to the hotel first: Dan or Tal? | |
| 4.1 Table-based<br>4.2 Graphical<br>4.3 Area-based | 4.4 Logical<br>*If Dan walks half the time at speed $v_1$ and half the time at speed $v_2$ and $v_1 > v_2$ then during the first half of the time he walks a longer distance that during the second half of the time. Thus he walks at the faster speed $v_1$ a longer distance than Moshe. Dan gets to the hotel first.*<br>4.5 Diagram |

Figure 1: Four MSTs from the test reported in this paper.

During the past two decades, I have systematically implemented multiple-solution tasks (MSTs) as a didactical and research tool in the majority of the studies that focus on the identification, development, and role of creativity in the teaching and learning of mathematics to students and teachers. During the last decade these studies have mainly analyzed creativity-related differences in learners with varying levels of excellence in school mathematics or in teachers with varying levels of expertise. Insight-based solutions in MSTs are in contrast to learning-based solutions of the problems

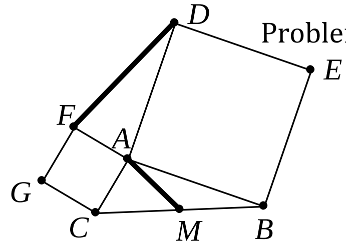
**Developing expertise by means of creative mathematical activities**

*I cannot think of even one [solution] and you ask [me] to produce two....*

*A mathematical challenge is an interesting and motivating mathematical difficulty that a person can overcome* (Leikin, 2007). Mathematical challenge is a core element of mathematical instruction aimed at fulfilment of the learners' mathematical potential through integration of cognitive demand (Silver & Mesa 2011) and positive affect (joy, interest and motivation) in the learning process. Jaworski (1992) stressed importance of mathematical challenge when defined teaching triad which characterises expert teachers by ability to combine mathematical challenge, sensitivity to students and management of learning in every instructional situation.

Challenging mathematical tasks can require solving insight-based problems, proving, posing new questions and problems, and investigating mathematical objects and situations. Investigation tasks (ITs) are the most inclusive (and thus the mist challenging) type of tasks directed at conjecturing, examining the conjectures, proving, and posing new questions (Figure 3). MSTs and ITs are unique since they are challenging for novices and experts alike as *creativity-directed tasks* since their solutions require flexibility when finding additional solutions and raising different conjectures as well as originality when finding new mathematical facts and new mathematical solutions.

The analysis presented herein is based on a case study that focuses on the individual and collective solution spaces and spaces of discovered properties of Prospective Mathematics Teachers (PMTs) who are considered (in the present study) non-experts in geometry problem solving. The PMTs participated in 56-hour courses directed at the development of their knowledge of geometry, problem-solving expertise and the ability to create new geometry problems through investigations in DGE (see also Leikin, 2014). ITs and MSTs, which served as a core element of the courses, were completely new for the PMTs at the beginning of the course. The sessions with PMTs were videotaped and artefacts of their works were collected. Additionally, the PMTs presented their investigations to the whole group of PMTs and these presentations were also video-recorded. We analyzed (Leikin & Elgrably, 2015) the PMTs' solution and discovery spaces using the solution and discovery spaces of Sharon (pseudonym) who is an expert in solving geometry problems at a very high level.



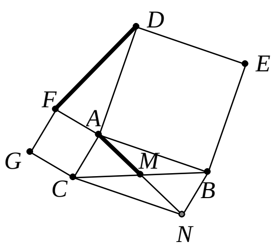

Problem 5: Two squares CGFA and BADE are built on the sides of a triangle ABC.

Prove that median AM in the triangle ABC is half of segment FD.

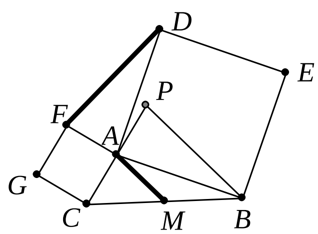
a. Prove the property in at least 2 different ways
b. Find at least 3 additional properties of the given object
c. Prove each discovered property in at least 2 different ways.

*Proofs*

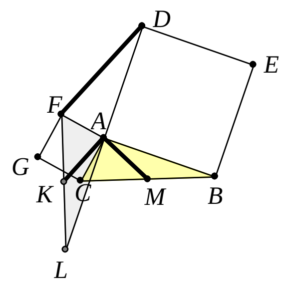
1. Doubling the median AM



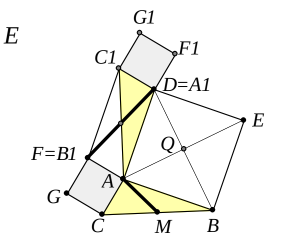

2. Doubling the side CA




3. Rotation of the triangle ABC by 90º about point A



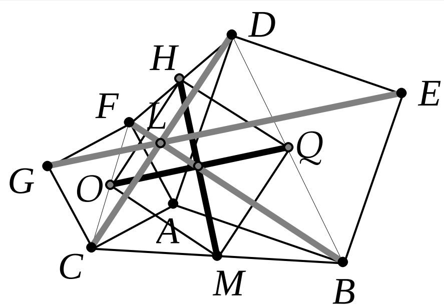

4. Rotation of the squire ACGF by 90º about point $Q$ (center of he squar $ADEB$)




*Additional properties discovered through proving*

1. $\Delta AFD \cong \Delta BND$(Proof 1); 2. AM ⊥ FD (Proof 3); 2. CB = 2AR (symmetry considerations);

*Properties discovered through investigations*




$O$- center of the square $ACGF$
$H$ -midpoint of $FD$

1 (a, b, c, d). Pairs of segments $FB$ and $CD$, $OQ$ and $HM$, are perpendicular and equal

2. $OHQM$ is a square
3. $BF \cap CD = L$, $L \in GE$
4. $GE \parallel OQ$
6. $GE^2 = FB^2 + CD^2$

6. $R \in AL$, $RV$ *and* $RW$ tangent to circles that subscrive the squires, then $RV = RW$

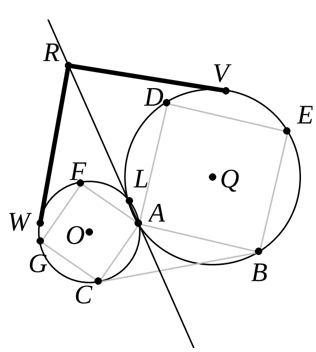
$CBST$ - square, $U$ center of the square $CBTS$

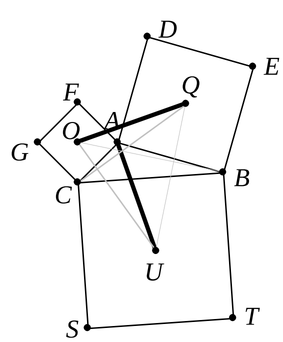
7 (a,b,c,d,e,f). Pairs of segments $OQ$ and $AU$; $OU$ and $CQ$; $QU$ and $OB$ are perpendicular and equal

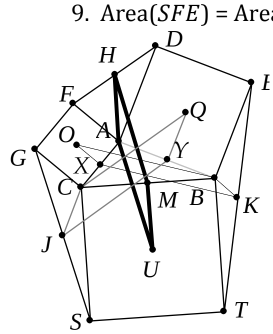
$X, Y, J, K$ midpoints of $CA, AB, GS, ET$

8. $AHMU$, $XOBK$, $CQYJ$ are parallelograms

9. Area($SFE$) = Area($GDT$)



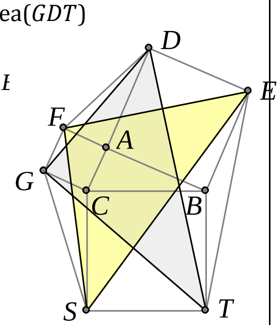




Figure 3: Collective solution spaces of solutions and discoveries of the investigation task produced by non-experts after training

The *discovered properties* were analyzed by all the participants according to: *(a) the newness of the property* discovered in the course of investigation, (b) *the complexity of the auxiliary constriction* performed for the investigation and (c) *the complexity of the proof* of the new property. We identified seven *types of discovery strategies* (1) Discovery by chance (through measuring in DGE), (2) Discovery through association with another problem; (3) Discovery in the search for a proof; (4) Discovery based on previous knowledge of related properties; (5) Discovery in the course of proof; (6) Discovery by symmetry considerations; and (7) Intuitive discovery.

The comparison of the expert and non-expert spaces emphasized the *relative nature of the discoveries*. Many of the properties discovered by the PMTs were deemed "trivial" by Sharon (since they "can be proved in two stages") or "familiar" (since "this is a common problem that appears in the textbook"). We also found that the experts' spaces of the discovered properties were usually richer; however, some of the properties were discovered by the PMTs and overlooked by Sharon. The major difference between expert and non-expert investigations was in the investigation-strategies applied. Most of the discoveries by PMTs were found "by chance" whereas Sharon exhibited a variety of conscious strategies.

Based on this teacher education experiment, we argue that problem-solving expertise is a core element in the development of investigation skills in geometry and in advancing PMTs' creativity. Figure 3 demonstrates that, collectively, PMTs produced 4 different solutions to the given problem and formulated more than 10 new problems (requiring proof of discovered properties) most of which were cognitively demanding. I invite readers to prove the discovered properties depicted in Figure 3.

I would like to stress the importance that collective solution and discovery spaces play in the development of individual solution spaces, that is of the PMTs' problem-solving expertise and flexibility. The affective component that accompanied the advancement of PMTs geometry knowledge and problem-solving expertise was evident in the change that occurred in the questions the PMTs asked:

From: Why make life so difficult? I can hardly find one solution while you are asking for two. If I could, I would leave the course. It’s unfair to demand these things of us.

How can I find a problem that has three solutions? I can see that different people can have different solutions. But how can I do it alone?

To: First of all, it's fun. At some point you feel you enjoy it -- enjoy solving and enjoy knowing. You say, "Wow, I can do it!". ...Why didn't we learn this way in school?

First, you have to force yourself to stop looking for which theorem the problem relates to; otherwise, you are confined to the theorem. Later you realize you are not interested in knowing on what page the task appears in the textbook. You simply solve it.

> When solving the tasks in the group, one is always surprised by how differently people think. We always had at least 3 or 4 solutions for a problem. So you start to believe that this really is possible.

The relative nature of the expertise also can be seen clearly in the question raised by one of the best students at the end of the course: *"How do mathematicians discover properties without DGE?"*

## SUMMARY

Following the literature review and based on our studies I agree that creativity can be fostered in the majority of students in a creativity-directed learning environment. Mathematical flexibility is a dynamic personal characteristic that can be considered as a function of expertise mathematical, whereas originality (as related to mathematical insight) seems to be a more innate characteristic and can be seen as function of the combination of mathematical expertise and general giftedness. Moreover, learning environment directed at promoting students' creativity is also effective for the development of their problem-solving expertise. Creation of such a learning environment requires teachers to be creative experts in mathematics teaching.

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# CREATIVITY AND IMAGINATION IN MATHEMATICS

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***Abstract.*** *Much research from around the world suggests that creativity can sometimes be viewed as an unwanted distraction from the aim of the lesson (Beghetto, 2007; Kennedy, 2005). Teaching mathematics without providing for creativity denies all students the opportunity to appreciate the beauty of mathematics and fails to provide students an opportunity to fully develop the understanding of mathematical concepts. The purpose of the article is to discuss a few factors that allow for students to experience creativity and imagination in the mathematics classroom. The discussion will be structured within a framework derived from recent research. It appears that most of the researchers' contribution to the theory can be categorized into two main sources of creativity, the internal and external factors. In this article much emphasis is given on the concept of imagination as the corner stone of creativity. Thus, we describe ways in which imagination can be developed by looking at problems through different lenses. To do this, we provide examples of tasks that have the potential of promoting students' mathematical imagination focusing on the process of bringing mathematical ideas to life through reframing problems, asking questions, connecting and combining, challenging assumptions, changing perspectives and representing ideas (Friedlander, 2016; Leikin, 2014; Leikin & Pitta-Pantazi, 2013).*

Creativity is a slippery concept. It lacks a universally accepted definition (Mann, 2006; Collard & Looney, 2014) and although there are numerous definitions for creativity, we will focus on the notion that a creative act is one that is new to the self and therefore allows the possibility that everyone is capable of creative activity to some degree (Boden, 2004; Craft, 2003). Thus, creativity may be defined as the generation of ideas. Being creative is to be able to generate or to come up with ideas. Thus when a child thinks of an idea, even if it is a naughty one, s/he is being creative. Similarly, when a scientist seizes on an idea, s/he is exercising creativity. Creativity is hence a process, or a thinking process to be exact.

Research into the field of creativity suggests that the current way in which the teaching of mathematics is approached only serves to stifle creativity (Bolden, Harries, & Newton, 2010; Kennedy, 2005; Mann, 2006; Skiba, Tank, Sternberg, & Grigorenko, 2010), and creativity can sometimes be viewed as an unwanted distraction from the aim of the lesson (Beghetto, 2007). On the other hand, there is a strong support to the claim that teaching mathematics without providing for creativity denies students the opportunity to appreciate the beauty of mathematics and fails to provide students an opportunity to fully develop their understanding of mathematical concepts (Mann, 2006).

The purpose of this article is to discuss a few factors that allow for students to experience creativity and imagination in the mathematics classroom. One of the most persistent questions in mathematics education is to find ways for students to come up with ideas and how students can make them happen. What students need is to open up the lens and look at the creative process and the process of bringing mathematical ideas to life through very different perspectives. The discussion will be structured within a framework derived from recent research (Seelig, 2012). It appears that most of research contribution to the theory of creativity can be categorized into two main sources, the internal and external factors as shown in the following diagram.

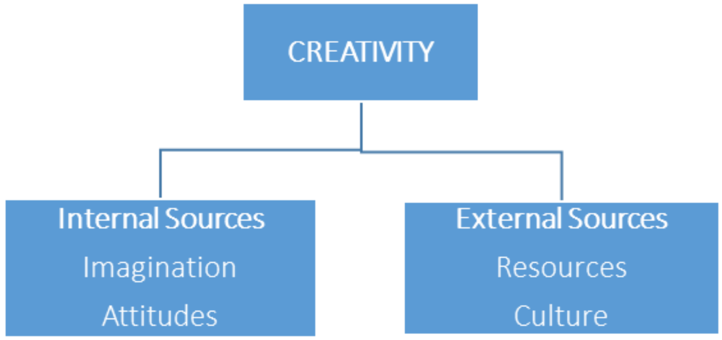



Diagram 1: Creativity, internal and external sources

Looking at the diagram, it seems that creativity is affected by the interplay between our inner mind (imagination, attitude) and our external environment (resources, culture). A number of researchers worked with these concepts in isolation, as pieces of a more general idea, but most of them imply or suggest that there must be some unified field theory to the creative process, not just a laundry list of things to do (Seelig, 2012). Following, is the description of the framework starting first from the internal sources. After the discussion of the external sources, a synthesis of the sources is provided. Throughout the discussion, examples of tasks which can infuse imagination and hence creativity in the mathematics classroom are provided (Bennevall, 2016).

## INTERNAL SOURCES OF CREATIVITY

### Imagination

Looking at the framework the first element of the internal sources of creativity is imagination. Much research on creativity starts with the role of imagination in mathematics education, and most importantly the discussion focusses on how can education foster imagination and creativity. The prevailing assumption is that certain habits, behaviours and strategies associated with imagination can be modelled in classroom learning (Craft, 2001). The mathematics curricula should promote and sustain environments *for* imagination and the *processes* for imaginative and creative thinking can be exercised in classroom teaching (Hoyt, 2002). To do this, teachers need to frame and reframe problems, to ask appropriate questions, to combine and connect mathematical ideas, to change assumptions during problem solving, to represent ideas and change perspectives for understanding and extending mathematical problems.

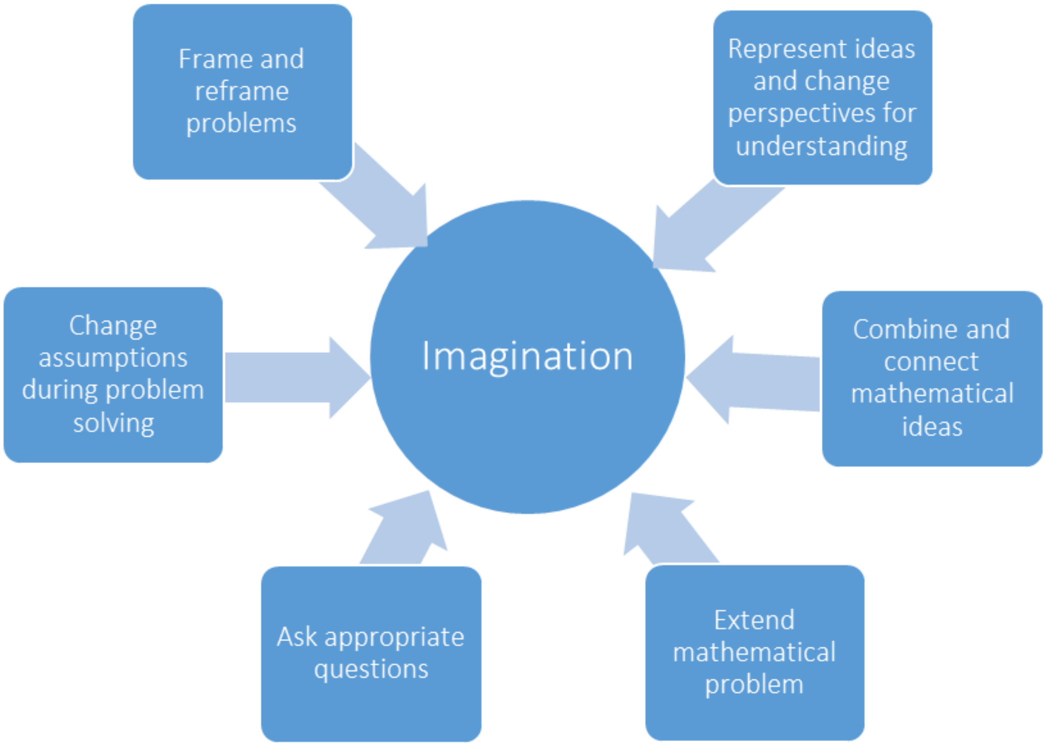



Diagram 2: Strategies for building imagination in the mathematics classroom

*Framing and reframing problems:*

We are all imaginative when we are kids. We come up with really interesting things when we are in kindergarten but over time we start seeing that creativity and our sense of confidence for our creativity dwindling. One of the reason for this, is the kinds of questions students are asked. Unfortunately, our educational system, starting in kindergarten, is built around asking students questions that have only one right answer. Yet, in the outside world most problems we encounter are open-ended. This is profound because the way you frame the problem determines the answers you will get (Morgan, 1993).

Creative people do not look at the world like this. They look at problems through different lenses and reframe the problems. For example, instead of teaching mathematics in the form of what is the sum of 10+10, teachers can ask a question like "Which two numbers add up to 20?", "Can you find other numbers that add up to 20?", "How many answers can you find that make 20?". This is critically important because often the answer is baked into the question asked and if you do not question the questions you are asking you are not going to come up with really innovative solutions. In fact, all questions are the frame into which the answers fall. Thus, we create frames for what students experience, and they both inform and limit the way students think. By changing the frame, teachers dramatically change the range of possible solutions (Morgan, 1993).

*Asking appropriate questions:*

The ordinary mathematics classroom begins with the answer and never arrives to real questions. In the ordinary mathematics classroom there is no room to doubt or to challenge answers or questions. The following is an investigation which mostly raises questions rather than provides answers (NRICH, 2017).

Let us start off with 1, 2, 3 and 4. Using the consecutive numbers chosen, we investigate what happens if we use different combinations of + and – on these numbers. We might begin in the following way:

$$1+2+3+4=10$$
$$1-2+3+4=6$$
$$1+2-3+4=4$$
$$1+2+3-4=2$$
$$1-2-3+4=0$$
$$1-2+3-4=-2$$
$$1+2-3-4=-4$$
$$1-2-3-4=-8$$

Diagram 3: Combinations of four consecutive numbers.

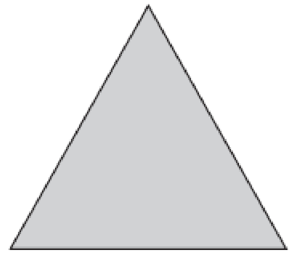
Looking at the calculations, students pay attention and have the opportunity to raise a number of questions such as "Why all the answers are even numbers", or "Why there is no calculation that gives – 6", etc. In addition, this kind of investigation provides the opportunity for students to change the consecutive numbers, say to 2, 3, 4 and 5 and observe whether the answers provided within the first set of consecutive numbers are also true for the new set.

*Changing assumptions*

Another way to increase students' imagination is with challenging assumptions. It means teachers going beyond the first right answer that students provide and assigning students tasks that are really surprising or where they need to come up with interesting solutions. Ponte (2007) highlights the opportunities for imaginative thinking provided by investigations as well as by problem solving. The following example is an investigation to illustrate the creative thinking allowing students not only to pose their own questions but also to move beyond the first correct answer.

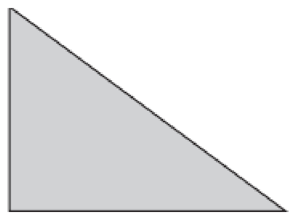
> "Consider triangles with integer sides. There are 3 sides with perimeter 12 units. Investigate."

There is a question lurking in the above situation, "What is going on with the sides of the triangle?". Intuitively, it feels like there is some connection between the lengths of the sides of the triangle. The first step students need to take is to pose the question and clarify the question. They know the context of the investigation (they are looking at triangles) but they need to pose their own problems to give them a start. They may ask whether they can draw such a triangle as described above:



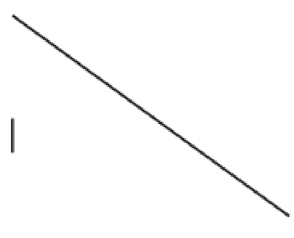
Triangle with each side 4 units in length

Even now the meaning of the lengths is not clear. The question feels authentic and compelling, like so many authentic mathematical questions. Thus, they need to go further and beyond their first answer by wondering whether they can draw another triangle that meets the conditions i.e. a triangle with perimeter 12 units.



Triangle with sides 3, 4 and 5 units in length

The next question they might ask is whether there are combinations of unit lengths that cannot form a triangle. In this situation they might try the extreme combination of sides 1 unit, 1 unit and 10 units.



In the above example, we can see that this simple mathematical context provides a rich opportunity for children to explore triangles. Linking to the idea of creativity, we emphasised that creativity involved both connecting to new areas or questions, and also reflecting upon questions that lead to the process of challenging assumptions.

*Changing perspectives and representations:*

The essence of imagination is the ability to change perspectives. Mathematics is about finding patterns and with a pattern we mean connections, some structure, some rules which govern what we see. Mathematics also means the ability to represent these patterns with a language. We construct a language if we do not have it. It is also essential to make some assumptions and play around with these assumptions to see what happens. If you change the perspective, you take another point of view and learn something new about what you are watching, hearing and looking. For example a simple equation like this one $\chi+\chi = 2\chi$ is also a very nice pattern. It is something we add that equals to something else and this is a beautiful perspective. There are two different perspectives. One perspective is a sum and the second one is a multiplication.

The same happens with any number in mathematics and students need time to see the patterns and the perspectives open up with representing numbers in different ways. For instance, most of the students know that the fraction 4/3 equals 1,3333333… . This representation is a pattern and it is written in base 10. Students could go further and play with different bases and thus write the number 4/3 differently. All these are representations of the same number and students can observe the different patterns in each number base (Antonsen, 2017). Furthermore, students could visualize 4/3 as a ratio or as rectangle with dimensions 4X3 and connect it with the screen of the average computer which is 800 X 600 or 1024X768 or 1600X1200. Students can also visualize 4/3 in numerous other situations such as in the formula of the volume of the sphere (Antonsen, 2017)

**Attitudes**

So to have imagination, students need to reframe problems, change the perspectives of problems, have knowledge, and pose questions that foster creativity. In addition to these, there is another important point of creativity and that is attitude. If students are not driven, motivated, have confidence that they can solve a mathematical problem, they will not solve it. Seeling (2012) in her book "inGenius: A Crash Course on Creativity", stresses the importance of attitudes in creative thinking, which is similar to the need for cognition or motivation. Teachers can do a lot in the classroom to develop the attitude for creativity. However, the most neglected factor for developing the attitude for creativity is time. It is not uncommon for students to graduate from high school believing that every mathematical problem can be solved in 30" or less and if they do not know the answer they are just not good in mathematics. This is a failure of education. We need to teach students to be tenacious and courageous to persevere in the face of difficulty. The only way to teach perseverance is to give students time to think and grabble with real problems. If teachers bring this image into a classroom and students take the time to struggle, they will realize that the longer time they spend the more the class comes alive with thinking.

Imagination and attitudes are part of the internal factors of creativity. They interact with each other and most of the time imagination is completed by the attitude to be creative and attitudes spark imagination.

## EXTERNAL SOURCES OF CREATIVITY

Resources are one of the external sources of creativity and are quite complex and multidimensional. Resources include among others the materials, the time, the technology and the appropriate problems that teachers have in their disposal to help students develop their creativity. But one of the things teachers often do not pay attention to is the physical space, which is critical for the creation of an environment in which creativity can flourish. Students need space where creativity is welcome, space where fun and ideas are welcome.

Culture is another important factor and is something that infuses entire school. Culture affects the way we feel, the way we think, the way we act and thus we need to think about it.

## EPILOGUE

The external and internal sources of creativity were described in short but the most important thing is to see the way in which the framework works, all in concert. The reason is that none of these sources stand alone. They all affect each other and without one of the others they cannot function. For example, imagination and resources are parallel, because the resources teachers create extend the manifestations of students' imagination. The recourses teachers build affect the way students think and the way they act certainly affects their imagination. This is true with attitudes and culture. Culture is the collective attitude of individuals and every individual contributes to that culture. The important thing is that teachers can start anywhere on this framework. There is no beginning and there is no end. The fact that creativity is so powerful and we as teachers have to switch it on.

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# RESEARCH AND PROJECT REPORTS

# ENHANCING CREATIVE PROBLEM SOLVING IN AN INTEGRATED VISUAL ART AND GEOMETRY PROGRAM: A PILOT STUDY

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***Abstract.*** *This article describes a new pedagogical method, an integrated visual art and geometry program, which has the aim to increase primary school students' creative problem solving and geometrical ability. This paper presents the rationale for integrating visual art and geometry education. Furthermore the MathArt pedagogy and program is described and it is explained how the MathArt program intends to increase students' creative thinking and geometrical ability. Additionally initial results of the pilot study are presented, which investigates the effects of the MathArt program.*

*Key words: creative problem solving; visual art; geometry; primary school*

## INTRODUCTION

In current primary mathematics education, there is little room for creative problem solving. Teaching materials in primary math education often focus on solving word problems in which students need to select and perform mathematical operations (e.g. Jansen, Van der Schoot & Hemker, 2005). Students learn to solve routine problems (exercises), but often do not learn to solve mathematical (non-routine) problems, which requires creative thinking because the student has no learned solution to solve the problem (Leikin & Pitta-Pantazi, 2013). Furthermore, teachers are often not used to teach for creativity. To change current math educational practices in primary schools, the MathArt project was started. It is a practice-based research project, in which researchers, primary school teachers and teacher trainers collaborate to develop a MathArt program for primary schools.

### Aims of the MathArt program

The MathArt program has the aim to increase students' creative problem solving skills in geometry and visual art and to increase students' geometrical ability in the upper grades primary school. To achieve these goals on a student level a teaching sequence for fourth, fifth and sixth grade students was designed in which geometry and visual art are integrated. Since it was expected that Dutch teachers were not sufficiently equipped to support students' creative problem solving in geometry, a professional development (PD) program for teachers was designed. This study has the aim to evaluate the effects of the teaching sequence and the PD program. In this paper the preliminary results of the pilot study will be presented.

## THE RATIONALE FOR INTEGRATING VISUAL ART AND GEOMETRY EDUCATION

The MathArt program focuses on geometry education, which is a good context to enhance student's creative problem solving, since it is never fully based on algorithmic procedures and therefore involves heuristic reasoning (Levav-Waynberg & Leikin, 2012). In this program geometry education is integrated with visual art education. Currently visual art and mathematics are taught in fixed disciplines and to solve a geometry or visual art problem, students might only rely on subject-related knowledge and are less able to 'break out' their thinking rut. We assume that integrating both mathematics and visual art, might help students to think in a more flexible and non-fixed way, and thus can enhance their creative thinking. Furthermore, the integrated context might also help the teacher to break out of their fixed idea of teaching math education in a separate discipline; it could show them that geometry education can also be taught in relation to other contexts instead of teaching geometry only from a math teaching method.

Creative problem solving plays both a role in mathematics and in visual arts. Several subprocesses can be distinguished that are important for the process of creative problem solving. Although diverse models of creative problem solving in general (e.g. Treffinger & Isaksen, 2005) or problem solving in mathematics (e.g. Schoenfeld, 1985) or visual art (e.g. Cawelti et al., 1992) exist, most models consist of several similar or overlapping subprocesses or stages. It is expected that subprocesses important in visual art might also help creative problem solving in geometry. An example is the role of orientation (defining the problem and recalling (pre-requisite) information related to this problem; e.g. Getzels & Cskszentmihalyi, 1975). Orientation seems to be important for creative problem solving, because if the problem and its related concepts are clear, it will be easier to produce a new and meaningful solution. Although the subprocess of orientation is also important in math education (SLO, 2008), it might play a bigger and more significant role in visual art (Getzels & Csikzentmihalyi, 1976) and visual art education (SLO, 2015). Within the Dutch visual art pedagogy, students need to learn to give meaning to a problem by reacting on it with association and memories; it is important that students sense (see, hear, feel) and talk about the theme and assignment from several perspectives during visual art education (SLO, 2015). In math it often seems to be more related to framing the problem, but not to recalling information necessary to understand the problem situation (SLO, 2008). A greater focus on the subprocess of orientation might also help geometry education. This is one example of how visual art might can give an impulse to geometry education.

## THE MATH-ART PROJECT – A BRIEF DESCRIPTION

The MathArt program consist of a teaching sequence for fourth, fifth and sixth grade students and a PD program for their teachers (see Figure 1), in which both the disciplines of visual art and mathematics are both equally covered and honored.

### PD program

The PD program for teachers consist of 5 sessions (each 2,5 hours), given by experts in the field of mathematics and visual art. After each session, teachers have to give one or two lessons of the MathArt teaching sequence. The aims of the professional development program are to learn teacher how to stimulate students' creative thinking in this integrated visual art and mathematics program, to create a positive attitude of the teachers towards geometry, visual art education and the integration of both and to increase teachers' geometrical knowledge and their pedagogical content knowledge of teaching geometry and visual art.

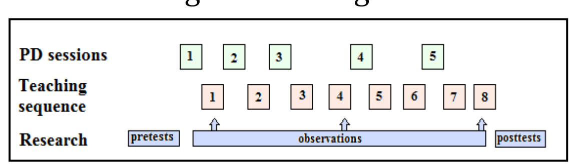


*Figure 1.* The MathArt research project

## The teaching sequence

The teaching sequence consist of eight lessons; 4 lessons related to the theme 'space' and 4 lessons related to the theme 'patterns'. Each lesson takes about 60 minutes and starts with an introduction on class level which takes between 15- 25 minutes. This is followed by an individual or group assignment that takes 25-30 minutes. A lessons ends with a reflection on class level which takes about 10 minutes. Aims of the teaching sequence are to enhance students' ability in geometry and visual arts and to enhance their creative problem solving skills.

## THE MATH-ART PEDAGOGY

In this section we will shortly report on the MathArt program, by describing the most important features of the pedagogy. Each lesson starts with **visual art reception**, with a groupwise discussion to activate children's prior knowledge and to develop students' visual perception and spatial reasoning. This part of the lesson is also meant to orient on the subject and the problem used in the lesson.

Within the program **open problems** and multiple solution tasks are used, because these can enhance students' creative thinking (e.g. Leikin & Pitta-Pantazi, 2013) and learning in geometry and visual art. The open problems are related to themes of both geometry and visual art, like perspective or symmetry and the open problem follows visual art production (Rouches-Levano Kerr, 1995). In this way students will learn to visualize their experiences, which is an aim of art education, but at the same time also learn to order and organize these spatial situations (geometry education). Furthermore, during visual art production, students work with materials and form visuospatial and sensorimotor representations of their personal experiences, which can help them in thinking and reasoning about geometry (Nunez et al., 1999).

During the lessons, students are engaged in a process of creative problem solving by producing visual art. Several subprocesses are expected to play a role when they creatively solve problems related to both geometry and visual art, namely **orientation, idea generation, idea evaluation and execution**. These subprocesses are interactive, do not occur in a certain sequence and can have a cyclic character (Leikin, Koichu, & Berman, 2009). So although some of these subprocesses are more implicit, it can help to make them explicit to enhance students' creative problem solving. Except for orientation, not all subprocesses need to become explicit in every lesson. Within the sequence of lessons there is varied explicit attention for each of these subprocesses. Orientation plays a role in every lesson, since it is important in art reception. Furthermore, during the execution of the assignment, the teachers evaluate the process and the products of the students so far, which enables the students to make a new product, or to adjust or improve it.

**Interaction with peers** is important for students to compare their ideas, see ideas from other points of view which can enhance creative thinking (e.g. Beghetto & Kaufman, 2010). It also increase students' learning since they have to explain their thinking, get feedback, and get other points of view. Interaction evokes reflection, which enables students to reach a higher level of understanding. Within the MathArt lessons students are stimulated to discuss, exchange and communicate ideas with peers, since research suggests that it can enhance students' creative thinking and geometry learning.

**Reflection** is very important within this integrated pedagogy, since it can help students to make explicit what knowledge and skills they have obtained regarding geometry and visual art. Reflection on the followed process and final product together with the teacher and peers at the end of the lesson, can help students to make explicit their implicit knowledge and skills obtained during creative problem solving. Reflection could extend and modify the existing knowledge, since students have to clarify what was going on and what they have learned. Furthermore it could help students to get more insight into creative problem solving strategies and how they could might be reapplied (e.g. Chi, De Leeuw, Chiu, & Lavancher, 1994).

An example of one of these lessons is '' playing with perspective''. Teachers start with an introduction in which they discuss a few visual artworks.  The teacher discusses six artworks in which artists have played with perspective and viewpoints. Two examples are the artwork of Escher (see Figure 2) and a photo (see Figure 3). Questions that teacher could ask during this introduction are 'What's happening in this picture?', 'What do you see that makes you say that?', 'What more can we find?', 'How did the artist created this effect?', 'Can you tell something about the viewpoint of the artist and what could be the reason for this?' and  'How would the photo look like when they would have used another point of view?'. After the introduction, students have to make photos in which they create illusions by playing with perspective and point of view in groups of 3/4 students. After 15-20 minutes, students need to select their two best photo's. Figure 4 shows one of the photos that students created.  At the end of the lesson the selected photos and their process of making the photos is discussed. Questions that the teacher can ask are for example: 'What effect did you want to create?', 'What did the students do to create this effect?', 'What perspective did they use?' and 'Where would they stand if we draw a map?'. Furthermore students need to reflect on what they have learned.

*Figure 2.* M.C. Escher (1947) Another world

*Figure 3.* A photo in which is played with perspective

*Figure 4.* One of the photos created by the students in MathArt pilot study

In the pedagogy of our PD program active learning is important. Therefore interactive methods are used in the sessions. Teachers for example have to experience the MathArt lessons themselves, watch film fragments of other teachers and have to make a hypothetical learning trajectory. Afterwards, teachers always have to discuss and reflect on these activities. The content of the PD program is related to the classroom practice. Furthermore reflection on and experimentation with the MathArt lessons is important; it can support on-going learning and encourages change.

## THE RESEARCH PROJECT

In a pilot study we evaluated the goals of the MathArt teaching sequence and the PD program. Fourteen teachers from grade 4, 5 and 6 in four schools in Rotterdam, the Netherlands and their students participated in the MathArt program from September 2016 – January 2017.  To quantitatively evaluate students' creative thinking and geometrical ability several pre- and post-measures were administered, like a geometrical ability test, a geometrical creativity test and more general measure of creative thinking (TCT-DP). To evaluate the PD program a pre- and post-survey was administered and teachers were observed during lesson 1, 4 and 8.

## EVALUATION AND FINDINGS

In this paper we will present some initial findings of the pilot study. More systematic evaluations are scheduled and the results will be presented during the conference in April.

One of the aims of the MathArt program is foster students' creative problem solving skills. Unstructured observations of the lesson ''Playing with perspective'' suggest that students are able to create photos in which they show illusions. Although some examples of photos with illusions were presented during the introduction, students came up with new ideas. Furthermore, one student also indicated that making the picture was very difficult. So although they might have an idea of how to create such a photo, actually making the photo requires different skills. By thinking of ideas, trying out and evaluating the picture (did it create an illusion?), students practiced their creating problem solving skills. Observations also showed the role of the environment for creative problem solving. Students also seemed to inspire each other; one group came up with the idea to try to touch the roof by standing on the ground, and another group continued on this idea in a slightly different way.

Another aim of the MathArt program is to increase students' geometrical knowledge by using creative problem solving activities. Although we cannot yet present results of quantitative data, qualitative data suggests students' improvement of geometrical knowledge. During the observation of one of the lessons, teachers asked their students after the lesson what they have learned. Students answers indicated: ''to find the right position for the camera and the position of the students to create these effects'', ''that other students have to stand at a very precisely place, otherwise you do not get the effect'' and ''every viewpoint creates a different effect''. Furthermore, unstructured observations suggested that students started to use the new words they have learned during the presentation of their photo's; they used words like perspective, bird's-eye perspective, frog perspective, position/viewpoint and optical effect while they were reflecting on their process and product. However, learning gains of students might differ between classes. Differences might be caused by the geometrical vocabulary introduced and used by the teachers and the kind of questions asked by teachers to help students during the process of creative problem solving and to help them explain geometrical phenomena.

Our observations in the first and fourth lesson of the teaching sequence showed the significant role of the teacher for stimulating students' creative thinking and their learning in geometry. Although we presented the fourteen teachers with the same lesson description, the way teachers enhanced students' creative thinking and geometrical learning varied a lot. For example, it is important that a teacher behaves like a facilitator; generate open ended questions that can extend students' explorations and thinking during the process of creative problem solving (e.g. Begehtto & Kaufman, 2010). However, during the observation of the first MathArt lesson, we saw large differences between teachers: some were getting coffee and walked away during the assignment, other teachers sat behind their desk and observed their students now and then , while others walked around through the class, asked questions to students about what they were doing and asked them questions about how they could improve their product. During the observation of the fourth lesson this point of observation already improved; more teachers behaved like a facilitator.

The implications that emerge from this pilot study are that this study can give more information about the effects of creative problem solving in a classroom and also on how creative problem solving emerges in a primary school classroom setting. Furthermore, the study also sheds light on practical issues and challenges related to (the implementation of) enhancing creative problem solving in current primary school education.

## Acknowledgement

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# ‘CAN YOU CHALLENGE ME IN REASONING PROBABILITY, PLEASE?’

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***Abstract.*** *One of the main issues of teaching talented students is challenging them. The research literature suggests that digital educational games have many potential benefits for mathematics and statistics teaching and learning. One of their foremost qualities is the capacity to motivate, engage, and immerse players. It has been shown that educational games captivate students’ attention, contributing to their increased motivation and engagement with mathematics and statistics (Ke, 2008). The current article contributes to the emerging literature on game-enhanced statistics learning by exploring the capabilities of a learning environment that uses programming logic in a game setting. Based on challenging students to create their own games, we attempted to enhance students’ (aged between 8 to 13 years old) reasoning about probability by asking them to design a computer game for modeling probabilistic ideas. Students were introduced to the block-based programming language Scratch 2.0, and used it to create their own games. In this article we present the case of a talented 13 year-old boy who expressed many probabilistic ideas while he was designing and playing his game.*

*Key words: Statistics education, randomness, educational game, game design, Scratch*

## BACKGROUND OF THE STUDY

Gifted and talented students tend to master mathematical and statistical ideas quickly, and thus may have already acquired the content and concepts corresponding to their grade level. Acceleration to a math class at a higher-grade level may be the most viable option in this case (Lewis, 2002). However, acceleration should focus on developing conceptual knowledge rather than moving students through the same content at a faster pace, as gifted students find mechanical skills easy to pick up (VanTassel-Baska, 2004). The conceptual acceleration plan may include enrichment experiences, differentiation of instruction including pretesting and compacting the curriculum, flexible cluster grouping by topic or mathematics achievement, grade skipping in math, mentoring, and increased use of innovative technologies. Technological advances, in particular, have provided the opportunity to significantly increase the range and sophistication of possible classroom activities, and to create entirely new, challenging learning environments in mathematics education.

One promising approach explored in recent years is the potential for digital games (videogames, console games, phone games, tablet games, etc.) to capture the interest of all learners, including the gifted and talented ones, and to promote the attainment of important competencies essential in modern society (Lowrie and Jorgensen 2015). Studies have demonstrated that, in addition to providing an incentive for young people to engage in learning, games also have the potential to yield an increase in students’ learning outcomes (Kolovou, van den Heuvel-Panhuizen, & Köller, 2013). Although much of the research on the effectiveness of gaming on learning is inconclusive at this point, there are strong indications in the literature that appropriately designed and constructively used games can support experimentation with mathematical and statistical ideas in authentic contexts, and can be used as the machinery for engaging students in creative, problem solving activities (Lowrie & Jorgensen, 2015).

While digital educational games can provide a range of potential benefits for mathematics and statistics teaching and learning, high quality, developmentally meaningful, digital games for students are less common than hoped. There is a wide variability in content, scope, design, and appropriateness of pedagogical features, with many educational games including mediocre or even inappropriate content, being drill-and-practice, and focusing on basic academic skills rather than on high-level thinking. Nonetheless, some exceptional exemplars that can help create constructive, meaningful, and valuable learning experiences for all students, including the gifted or talented, do exist. One promising type is coding gaming software, which teaches students the concepts behind programming in a playful context. Several educational applications are currently available for helping students with no coding background or expertise to grasp the basics of programming through the exploration and/or creation of interactive games (e.g. Scratch, ScratchJr, HopScotch, Bee-Bot). Often, coding game applications enable students to share their games with others, and to play or edit games programmed by others.

This increased popularity and proliferation of computer games has led to a widespread interest in their use as learning tools. The present study showed some insights from students' reasoning about probability while they were designing Scratch games.

Based on a case study of a group of students (aged 8-13) who developed their own games through use of the visual block-based programming language Scratch 2.0 (Massachusetts Institute of Technology, 2013), the present study describes how a student, considered to be talented, uses elements of reasoning about probability when he designs his own game.

## METHODOLOGY

### Context and Participants

A total of four workshops were organized and each one lasted for 2 hours. Twenty-six students (N=26, 16 male, 10 female), aged between 8 and 13, participated in all four workshops during July 2016 (summer school vacations) on a volunteer basis. An invitation to parents was placed in social media and the students were selected from a priority list based on registration date. All participants had the right to pause or stop their participation entirely at any given moment. Additionally, all parents provided their written consent regarding the use and publication of their students' work for research purposes. In this paper, all names used are pseudonyms in order to preserve participants' anonymity.

To serve the role of the gaming platform, our research team chose Scratch, a visual programming language developed at the MIT Media Lab that consists of reusable pieces of code, which can easily be combined, shared, and adapted. Scratch can be used to program interactive stories, games, and animations, art and music and share all of these creations with others in an online community (http://scratch.mit.edu/). It was created to help students think more creatively, reason systematically, and work collaboratively, all of which are essential skills required for the 21st century (Resnick, 2007). For each workshop, a different set of extra-curricular activities were closely designed based on constructionism (Papert, 1980), and each meeting was structured in such a way as to promote an unhurried and creative process. Finally, students were asked to create their own game based on what they had learned. In the last session, students continued their games from their previous meeting, changed them if they wished, and asked a friend to play their game so to identify any bugs and fix them.

### Data Collection and Analysis

For the purposes of collecting our data, we used a variety of methods, including live video recording of the workshop and screen capturing of the participants' interactions with the software. Other sources of data also included field notes and classroom observations. In six cases, we also conducted individual mini-interviews of selected students (interviewed while engaging in game design) that expressed some exceptional ideas regarding the element of randomness, in an attempt to study further their contributions to this project. For the purpose of analysis, we did not use an analytical framework with predetermined categories. What we instead did was, through careful reading of the transcripts and field notes and examining of the various interactions for similarities and differences, to identify recurring themes or patterns in the data. To increase the reliability of the findings, the activities were analyzed and categorized by all three researchers and any inter-rater discrepancies were resolved through discussion.

## DESIGNING A SCRATCH GAME: A CHALLENGE FOR PROBABILISTIC REASONING

In the following paragraphs, we present a case of one student (out of the twenty six students who participated in the study) who is reasoning about probability in the context of creating his Scratch game. As the course was based on a volunteer basis, it was really important for us to accelerate the challenge of each student.

Chris, a 13 year-old boy, was one of the students who really liked using randomness in his games. This boy was a talented student, who was trying to build his own game. He was sitting at the back of the classroom and participated in class discussions only if he had to say something about his game. Chris designed a game of a dog crossing the street. The aim of the game was to help the dog to cross safely (without touching any of the cars).

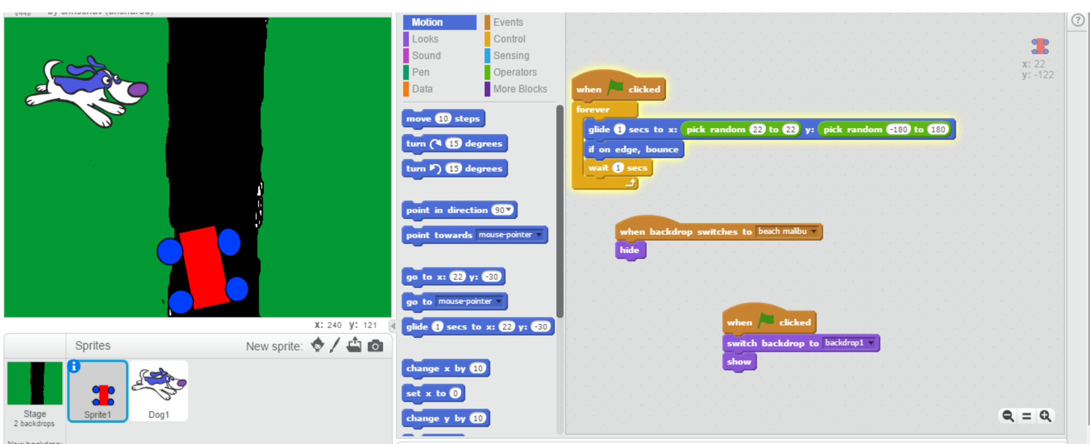


Figure 1: Chris' first version of random game

Chris described the way he built the game as follows:

1 R: Why didn't you just make the car to move forward?
2 C: This is boring…just seeing the cars and move around. Now you don't know…Of course it is easy with one car. …Chris is making different things on his game.

The reason underlying Chris's use of randomness in his game, was to make the game interesting. The challenge for him was not only to create a game by using randomness, but also to create a stimulating game. We found it interesting that Chris' game was a non-deterministic model of crossing a road. His idea of moving the cars in the road randomly is what makes his game appealing.

Chris designed a car that moved in a random way. Although a random movement of the car might have sufficed for the aim of the game, he also used the road as a spatial sample space and tried to increase the difficulty of the game by increasing the number of cars passing by. This also shows that Chris was not really 'happy' with his solution and he wanted to continue his game design. He moved his design to a second version.

3 R: What have you done?
4 C: I just put two cars, a counter, made a bigger road and I changed the dog. I changed the code of the cars.
5 R: Why?
6 C: It is better this way. I made the road bigger and I asked the cars to move randomly all over the road. This makes it more difficult for the dog to cross.

Chris judged his own first version of the game and he created a more difficult game. After playing the game again, he concluded.

7 C: Actually, that way is not interesting…it's not fair. You know…I can make some change to the design. I will make the dog smaller. That will make it fair…Let's see.

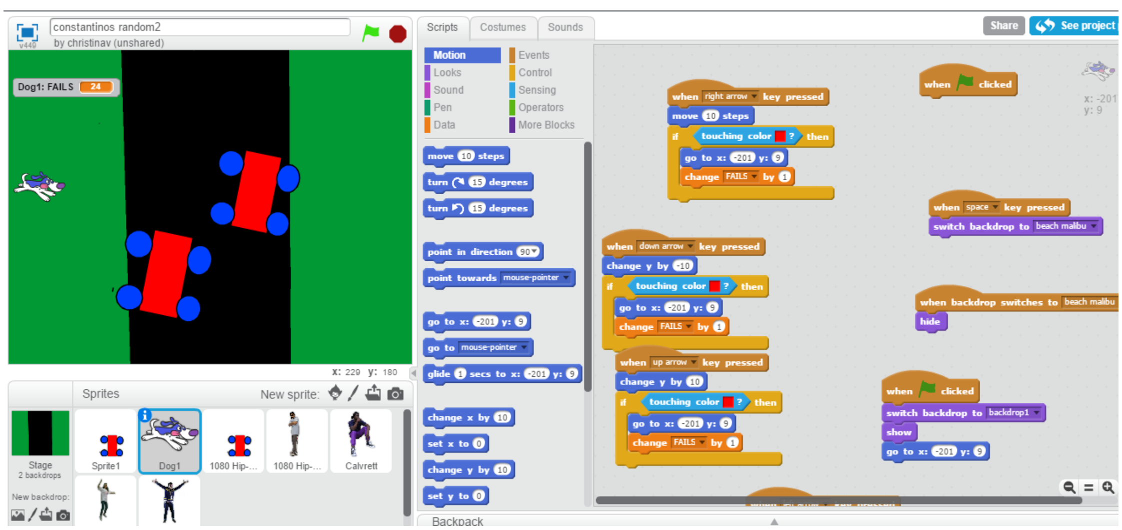


Figure 2: Chris' final version of random game

Chris used the idea of fairness and the probability of 1/2 in his game while he was designing and re-designing his own game. It is interesting that although in the workshop we never referred to spatial probability, Chris in his game connected the concept of space with the concept of probability. We can see that he did not change the code in his game, although he could have done that in order to reduce the probability of the car crossing the road. What he did instead was to come up with the innovative idea of reducing available space in the road.

At the end of Chris' last version of game, we asked him if he was happy with that.

8 R: Do you think you can play this game with a friend?
9 C: Yes… Now, it works…
10 R: Why?
11 C: It is a fair game, you can win, but you have to take care… It is not impossible for the dog to cross the road, but you should make a strategy based on the cars movement…It is interesting like this, but if my friend wants me to make some changes, I might do them.

It was obvious that Chris was on the one hand happy about his game, but on the other hand felt the urge to change the game once more.

## DISCUSSION AND CONCLUSIONS

The increased popularity and proliferation of computer games has led to a widespread interest in their use as learning tools. Several statistics educators have, in recent years, been experimenting with digital games, investigating the ways in which this massively popular worldwide youth activity could be brought into the classroom in order to capture students' interest and facilitate their learning of statistical concepts (e.g. Pratt et al., 2008; Paparistodemou et al., 2008; Meletiou-Mavrotheris, 2013; Erickson, 2014). The aim of the research reported here was to explore, through a case study of a 13-year old talented student, how a gifted or talented student might use elements of reasoning about probability while designing his/her own game. The case study of Chris, shows in practice how appropriate software programs offer opportunities for talented students to advance at their own rate. Our study findings indicate that randomness is an important factor to consider when designing and playing games, and that a software like Scratch can provide opportunities to fill the gap between intuition and conceptual development of probabilistic ideas (Batanero & Diaz, 2012). When we reconsider prior work on randomness (for example, Pratt, 2000), we find resonance in the use of symmetry between apparent fairness and the tendency for children to consider the appearance of the dice (or coin, or spinner…), something that we also found in Chris' case. Chris, because of designing, used simultaneously the idea of randomness in terms of the icons he used in his game. We were really surprised with how these ideas came out without even mentioning what sample space is, or how we calculate probability.

The design, coding, revision, and debugging of computer commands, helps students develop higher order problem solving skills such as deductive reasoning, while at the same time improving their conceptual understanding of key mathematical and statistical ideas. Thus, it becomes crucial to incorporate computer programming into existing statistics curricula. Game coding learning environments provide an ideal opportunity for doing so in an engaging, non-threating, and child friendly manner (Resnick, 2007). Educators and others can ensure that coding gives opportunities for new expressions, even for reasoning about probability. May be this is what the challenge can be for all the students.

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# CREATION OF MATHEMATICAL OBJECTS AS ASPECT OF CREATIVITY IN PRIMARY GRADES

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***Abstract.*** *In contemporary literature, there is a strong connection between mathematical creativity, problem solving and problem posing. However, the free or purposeful creation of mathematical objects can also be seen as an important aspect of mathematical creativity. In the first theoretical part of the paper, we will further elaborate the concept of mathematical creativity in primary school age. Typical examples of the yet neglected creation of mathematical objects are discussed in the second part.*

*Key words: mathematical creativity, creation of mathematical objects*

## CREATIVITY

Since the 1950s, creativity research has continually been and is still being advanced. However, to this day there is neither a consistent definition of creativity nor a commonly acknowledged creativity theory (e.g. Haylock, 1997). Instead, there are different approaches, or paradigms, that have been used to understand creativity (Sternberg & Lubart, 1999). Generally, creativity refers to the generation of *products* that are perceived – or valued – to be creative, or to a special kind of thinking *processes*. Based on that, creativity is also seen as a quality of a *person* and creativity-affecting environmental factors are discussed (*press*). Based on the work of Rhodes (1961, p. 307), these description efforts are also referred to as the "4 P's of creativity".

What is normally considered the key criterion of a creative product is its "novelty". However, since objective novelty independent of space and time is extremely rare, some authors relativize this criterion in so far as an idea is seen as new (or unique) if it is rare among a particular population (e.g. Guilford, 1967; Jackson & Messick, 1965). Furthermore, in pedagogic situations an individual reference norm might be used as a basis for the novelty criterion (e.g. Kießwetter, 1977), which is then referred to as individual creativity. This approach could be especially appropriate for primary school age.

Besides novelty, at least one further criterion is specified, which concerns the purpose of the product. In this regard, terms such as "meaningfulness", "target-orientation", "real-life relevance" and "usefulness" (Preiser, 1976) are used. It should, however, be mentioned that free creative processes would not be classified as creative according to this approach if the created products did not meet the criterion of usefulness. Since this would apply to many artistic products, the usefulness criterion might be seen as disproportionally constraining the kind and number of creative products.

Regarding creativity as a quality of individuals, the features based on the work of Torrance (1966) are usually cited, namely fluency, flexibility, originality and elaboration.

## MATHEMATICAL CREATIVITY ON AN INDIVIDUAL LEVEL

There is also a lack of an accepted definition of mathematical creativity (Haylock, 1997; Mann, 2006). The above-mentioned features of creative products and people are, however, also applied in mathematics. More precisely, mathematical creativity is described in the literature, for instance, as the ability to choose (sensu Poincaré), to engage in non-algorithmic decision-making or to generate novel and useful solutions to problems (Sriraman, 2009). Krutetskii argued already in 1976 that mathematical creativity in schoolchildren can be recognized in "the independent formulation of uncomplicated mathematical problems, finding ways and means of solving these problems, the invention of proofs and theorems, the independent deduction of formulas, and finding original methods of solving nonstandard problems." (p. 68). Also according to recent literature, creative mathematical processes particularly occur while problem solving and problem posing (e.g. Bonotto & Dal Santo, 2015; Leung, 1997; Silver, 1994; Yuan & Sriraman, 2011). Furthermore, we deem important that results of creative mathematical processes do not always have to be applicable, which is why "[…] it is sufficient to define creativity as the ability to produce novel or original work" (Sriraman, 2009, p. 15).

With a view to primary school, we can say that problem solving is already an important part of mathematics classes, whereby most typically used problems can be solved without a broad mathematical knowledge base. Since it is particularly challenging for young students to formulate their own questions, creativity in connection with problem posing might mainly manifest itself in the design of diverse variations of given, perhaps even partly solved problems.

With mathematically interested and experienced older students and even more with adult researchers in mathematics, problem solving and problem posing are usually embedded in more comprehensive *theory building processes*. Here, the handling of an initial problem – which has often to be concretized by the problem solver – becomes part of a circular process of problem solving and problem posing through variation and expansion and the subsequent analysis of this circle. The results and methods as well as the newly developed terms and logical relations and, respectively, the novel strategies and tools emerging from this process form a "theoretical fabric", which is then optimised, preserved and integrated into the existing knowledge base (Fritzlar, 2008).

In such theory building processes, creative acts, the invention of (subjectively) new mathematical objects as well as new mathematical methods play an important role. They also specifically highlight the interplay between divergent thinking, i.e. the ability to develop and elaborate diverse and original ideas with fluency, and convergent, i.e. logical and evaluative, thinking.

With a view on primary school, it can hardly be expected that students with little mathematical experience are already capable of such theory building processes. However, concerning mathematical creativity, it could be possible that primary school children are already able to *create subjectively new mathematical objects and gain mathematical experiences in investigating these*. Thereby the student's invention can either be rather target-oriented, especially when they are dealing with a superordinate problem, or rather free (in the sense of Sriraman mentioned above).

In the following figure, we have summarized important possibilities for prompting mathematical creativity with persons of different levels of expertise. (A comprehensive version can be found in Assmus & Fritzlar (in press). It also considers personality factors which cannot be discussed in this paper due to lack of space.)

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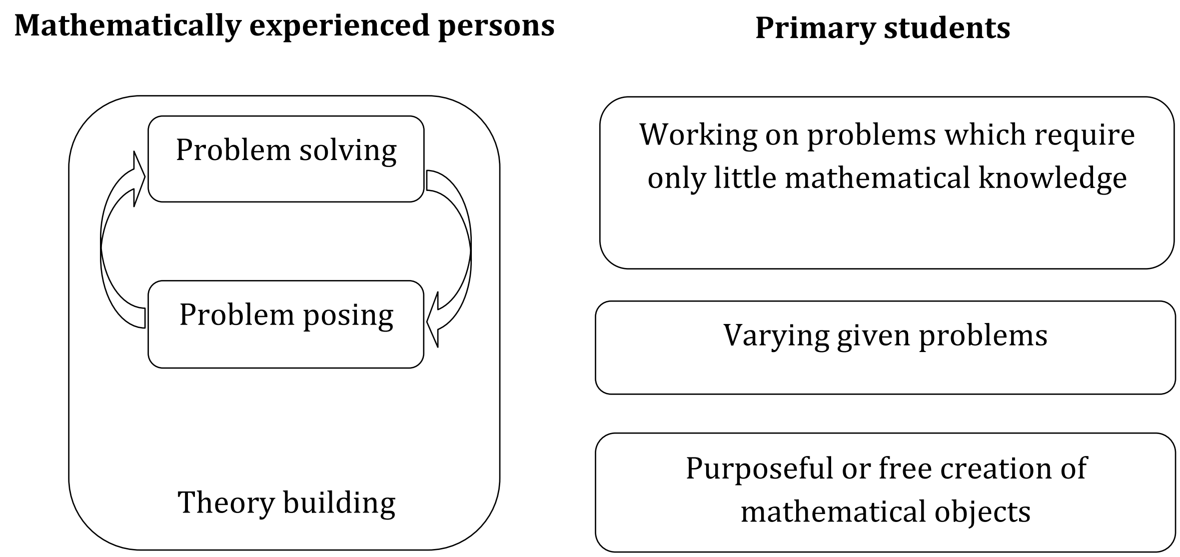



Fig. 1: Prompting mathematical creativity

## CREATIVITY AS INVENTING MATHEMATICAL OBJECTS IN PRIMARY SCHOOL

In recent mathematics education literature, problem solving, problem posing and their relations with creativity are intensively discussed. By contrast, the creation of subjectively new mathematical objects as an important aspect of mathematical creativity can be seen as a neglected research topic – at least for primary grades. However, creating new objects is already feasible for primary students if these processes are prompted accordingly and appreciated by the environment.

To begin with, mathematical objects to be created in primary school age can be categorized in the following way:

- Types of numbers (or rather sets of numbers with special properties)
- Types of shapes (or rather sets of shapes with special properties)
- Operations
- Algorithms
- Functions
- Formulas
- Arithmetical or geometrical patterns, arithmetic-geometric pattern sequences

First, this is to be concretized by an example in which students are inventing mathematical operations without being asked to do so. Paul, a mathematically interested third-grader attending our math circle at the university in Halle, developed a new mathematics textbook for the first grade on his own. He worked on the book together with two classmates whenever they were bored during math lessons.

On the first pages the authors introduced some new operations (with imaginary names) by means of a few examples, they continued with many exercises, tests and also with special tasks for mathematically strong and weak first-graders. Three operations are presented in the following figure.

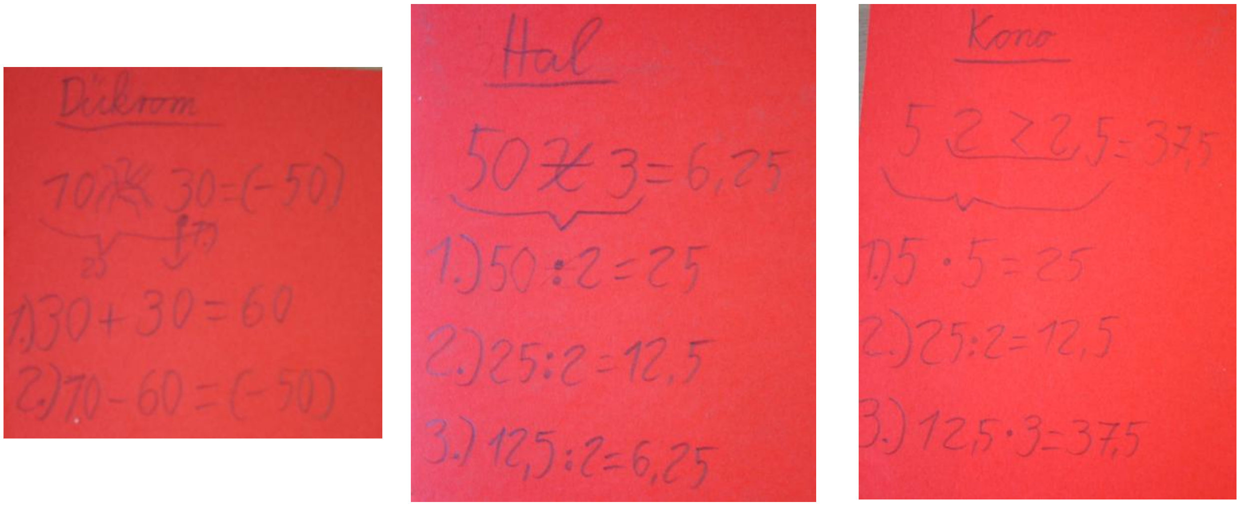



Fig. 2: Operations invented by the third-grader Paul

In the new operations, at least two computing steps or two known operations were combined. The third-graders already used decimal and negative numbers. By recording individual steps of computing, it becomes clear that the calculation was not always done from left to right. It also seems interesting that the second number in the operation HAL does not indicate what, but how often something is to be calculated; the idea of powers is already indicated at this point (Fritzlar, 2011).

In the many years we have been working with mathematically strong primary students, we could only very rarely recognize such extensive and autonomously triggered creative processes. Therefore, environmental factors initiating, supporting and appreciating mathematical creativity seem to be, in our view, very important.

Therefore, we constructed mathematical situations in which students first had the opportunity to get acquainted with the kinds of mathematical objects before they should invent their own. In the first phase, we set the following tasks for the various types of objects:

- Types of numbers: Recognizing and describing a common characteristic of a given set of numbers; specifying further numbers satisfying this characteristic
- Operations: Decrypting predetermined unknown operations, which could be described as combinations of already known operations
- Arithmetical patterns: continuing number sequences; describing the used rules
- arithmetic-geometric pattern sequences: continuing a given figure pattern, identifying the number of elements in the next and a later component; describing the used rules

In the following we will present some relevant experiences with third- and fourth-graders. For additional examples see also Assmus & Fritzlar (in press).

## Inventing types of numbers

Fourth-graders from a regular school class invented different sets of numbers with characteristic features. They partly focused on numerical properties, such as divisibility by 7 (Fig. 3) or on relationships between the digits of the numbers (Fig. 4 and 5). The example in figure 6 can be assigned to both categories: relations between digits (same ones-digit) but also numeric properties (remainder 3 in division by 10) are apparent.


Fig. 3: Divisibility


Fig. 4: same checksum

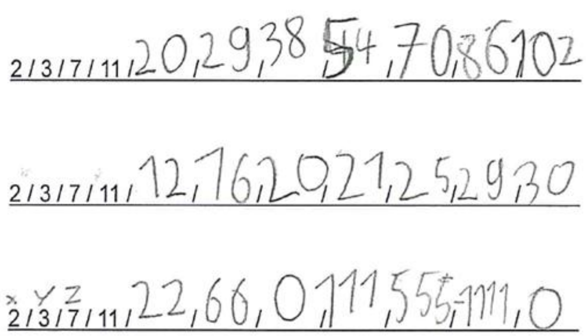

Fig. 5: palindrome of numbers


Fig. 6: Equality of digits

## Inventing arithmetical patterns (number sequences)

The invention of number sequences can refer to the continuation of a beginning sequence of numbers or to a complete new construction. The first variant is considered here. In a math circle meeting, fourth-graders were asked to find three different suitable continuations. The students' solutions showed a huge spectrum of different ideas. The use of geometric mappings such as "reflecting" numbers or "translating" a section of the sequence, and the systematic variation of the differences between adjacent numbers are only a few of them.

As an example of the wide range of continuations generated by individual students, one student solution is given in Figure 7. Especially the third continuation can be seen as very creative.




1. $+1^2, +2^2, +2^2, +3^2, +3^2, +3^2$
2. +1,+4,+4,+1,+4,+4,+1,+4,+4,+1
3. x, y, z, (x-1)·11, (y-1)·11, (z-1)·11, (x-2)·111, (y-2)·111, (z-2)·111, (x-3)·1111

Fig. 7: Three different continuations of the same number sequence — used rules

Due to lack of space, only a very small part of our extensive investigations could be presented here. All in all, they revealed that already primary school children are capable of developing subjectively new mathematical objects. Corresponding mathematical tasks are, in our view, well suited to foster mathematical creativity even in primary school.

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# EYE-TRACKING AS A TOOL FOR INVESTIGATING MATHEMATICAL CREATIVITY FROM A PROCESS-VIEW

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*Mathematical creativity with its increasing significance in our society and economy is more and more in the focus of mathematics education research. Previous research has focused, among others, on Multiple Solution Tasks (MSTs), where students aim to find many solutions to a given mathematical problem. Whereas most research using MSTs addresses students' products resulting from their creative endeavours (product-view), recent research has asked the question of how students' creative processes look like in which they come up and pursue their creative ideas (process-view). In this paper, we address this question focusing on eye-tracking (ET) as a research method; which has the potential to complement well-established methods in creativity research and reveal insights that may complement the body of research. We give an overview on the opportunities of ET for investigating mathematical creativity. In particular, we discuss the possibilities of different types of ET technology; and how ET using this technology can contribute to studying mathematical creativity.*

*Key words: Mathematical Creativity, Eye-Tracking, Eye Movements, MSTs, geometry, proof*

## INTRODUCTION

Creativity is significant for generating novelties, finding original ideas, or treading new paths of thinking. This is crucially important not only for mathematics but for all STEM areas (science, technology, engineering, and mathematics) in the increasingly automated and interconnected high-technology based societies and economies that we are and will be living in in (cf. OECD, 2014). We see a trend in mathematics education increasingly focusing on creativity: It is no longer sufficient for students to solve problems with routine schemes or familiar heurisms. Educators want to enable students to think "out of the box", to connect different topics whilst solving problems, to have "aha!" experiences.

Thus, research has addressed the question of how to foster and—as a prerequisite—study mathematical creativity. Whereas research has mainly focused on students' products—their written solutions and drawings (e.g., Kattou et al., 2013; Leikin & Lev, 2013)—, we see a trend in the recent scientific discussion to furthermore consider students' creative *processes* (e.g., Schindler et al., 2016) and inquire into the question of how students solve problems creatively; how creative ideas emerge, what triggers them, and how students can be supported in solving problems creatively.

Investigating mathematical creativity in a process view is, however, a challenge for researchers. Existing well-established research methods (e.g., thinking-aloud interviews, or videotaping students' creative problem-solving) have contributed to generate a considerable body of knowledge regarding students' mathematical creativity. However, we see that it is difficult to get "online" access to students' attention in their very creative work, because research methods are dependent on students' verbalizations, gestures, writings, etc.—on externalizations of students' creative processes. We see that ET offers a suitable additional research tool: With ET, where the movements of a person's eyes are "tracked" with cameras, researchers can study where exactly persons look at while, for instance, solving an MST. They can get closer to students' foci of attention when working creatively. Like Holmkvist et al. (2011), we think that "[t]here is no doubt that it is useful to record eye-movements, it advances science and leads to technological innovations" (p. 1)—also for research on mathematical creativity. In this paper, we elaborate on the value that ET may hold for investigating mathematical creativity; in particular students' creative processes when working on MSTs.

## MATHEMATICAL CREATIVITY

Today's view on mathematical creativity is predominantly influenced by Guilford's theory of intelligence (Guilford, 1967), which sees creativity as one dimension of intelligence. Guilford's theory emphasizes divergent thinking, the ability to find unique and manifold ideas, which he pictures as "most relevant to creative performance" (p. 169). Divergent thinking in this approach is conceptualized and evaluated in four dimensions: *fluency*, as the number of solutions; *flexibility*, as the diversity of produced solutions; *originality*, as the uniqueness of produced solutions; and *elaboration*, as the level of detail. Guilford's approach has largely been used in educational research and mathematics education research in particular (e.g., Kattou et al., 2013; Leikin & Lev, 2013). Here, researchers draw on Multiple Solution Tasks (MSTs); mathematical problems that can be solved in different ways. MSTs can be used both for evaluating and fostering mathematical creativity among students. Tests to evaluate mathematical creativity mostly draw on MSTs and refer to Guilford's categories of fluency, flexibility, and originality for scoring creativity.

Further ideas on mathematical creativity and its evaluation have emerged in the scientific discussion in mathematics education (e.g., Schindler et al., 2016; Liljedahl, 2013). Whereas most previous research focused on creativity as a product or on creativity as an attribute of a person, the idea to treat and evaluate creativity as a *process* (Rhodes, 1961) has increasingly attracted attention. Research with a process perspective on creativity mostly draws on Poincaré's (1948) and Hadamard's (1945) ideas, focusing on the processes of preparation, incubation, illumination, and verification (Sriraman, 2009). Contemporary research in mathematics education on creativity asks the question of how such illumination emerges and "what is the nature of this phenomenon?" (Liljedahl, 2013, p. 253).

Research investigating students' processes when working on MSTs links both theoretical strands: the theoretical assumption that finding many different ways of solving a problem reflects creativity (based on Guilford's ideas) and the assumption that the creativity process incorporates different phases, in particular incubation and illumination or insights (based on Poincaré's/Hadamard's ideas). Questions such as how original ideas come up, or what leads to the so-called Eureka!-moment when students are working on MSTs are the springboard for our research using ET.

## EYE-TRACKING (ET)

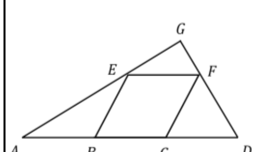
Eye-tracking (ET) is a technique to capture persons' eye movements when they are looking at stimuli at hand (Chen, 2011). It can capture—among others—participants' fixations and saccades, which are typically analyzed in ET research (Salvucci & Goldberg, 2012). *Fixations* are moments when the eye remains relatively still and focuses—consciously or not—stably on a certain focus point or a small area. *Saccades* are fast eye-movements in between fixations (Chen, 2011). Humans make—intentional or triggered by a reflex—about three of such saccades per second on average (Jang et al., 2014).

Research using ET technology has increasingly gained popularity over the last decade (Andrá et al., 2015; Salvucci & Goldberg, 2012) and is accessible more than ever before (Holmkvist et al., 2011). It draws on the so-called "eye-mind" hypothesis (Just & Carpenter, 1976) meaning that what a person looks at is in the focus of the person's cognitive processes and that a person's eye-movements are tightly related to their cognitive processes (Jang et al., 2014). Accordingly, eye-movements are understood to offer a "dynamic trace" of a person's attention and its shifts (p. 318). In particular, fixations indicate attention and cognitive processing of information; saccades hint at shifts of attention; together, they indicate how persons acquire information and what information they are attending to (Andrá et al., 2015; Jang et al., 2014; Chen, 2011). They even hint at persons' *intentions* and may predict their *future actions*: If a person, for instance, holds her hands under the water faucet and then fixates the soap, this presumably reflects the intention to use the soap and may (not necessarily) be followed by an action (the person taking and using the soap). However, the person can also drop this intention, e.g., when the person realizes that the soap dish is empty, and not carry out the action. In mathematics education research, different ET studies have been conducted focusing, e.g., on students' thinking modes, problem solving, strategies comparing fractions, etc. Regarding students' work on geometrical MSTs, previous ET research indicates, e.g., that highly creative people tend to have longer and faster saccades than less creative people (Muldner & Burleston, 2011, see below) and that ET overlaid videos provide detailed insight into students' creative processes (Schindler et al., 2016, see below).

## ET IN CREATIVITY RESEARCH

Relatively recently, ET has been applied for investigating mathematical creativity using geometrical MSTs (Schindler et al., 2016; Muldner & Burleston, 2015, Fig. 1). As mentioned above, ET holds the potential to gather additional insights into students' creative processes on MSTs and, thus, to contribute to the body of knowledge. In particular, geometrical MSTs are suitable because it is likely in geometry MSTs that the students' cognitive attention lies on the same focus that they are visually paying attention to.

Solve the following problem. Can you find different ways to solve the problem? Find as many ways as possible.

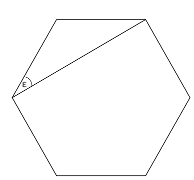



In triangle AGD, points E and F are on AG and DG respectively, and points B and C are on AD.

Given that EF = FC = CB = BE and AB = BC = CD, prove that triangle AGD is a right triangle.

**Triangle MST** (used in Muldner & Burleston, 2015)

Solve the following problem. Can you find different ways to solve the problem? Find as many ways as possible.




This figure is an equilateral hexagon. How big is the angle ε?

Remember, in an equilateral hexagon, all sides have the same length and all angles have the same size, which is 120°.

**Hexagon MST** (used in Schindler et al., 2016)

Figure 1: Multiple Solution Tasks (MSTs) applied in ET studies.

## ET setups and techniques suitable for creativity research

ET approaches determine the orientation of the eye ball and can be classified into three categories: contact lens, electro-oculogram, and video-based (Lupu & Ungureanu, 2013). The first two categories are strongly invasive—participants need to wear contact lenses with mirrors, or electrodes near the eye—and, thus, not suitable for creativity research with students. In video based approaches, the pupil is located in a video stream. For research on mathematical creativity, video based approaches are expedient since they are (relatively) non-intrusive. Most commonly, *static eye-trackers* or *head-mounted eye-trackers* are used (cf. Holmkvist et al., 2011). We see that both kinds of eye-trackers are suitable for creativity research; however, the choice depends on the purposes of ET studies, in particular on the used tasks, and on the available resources; as outlined below.

Among *static eye-trackers*, *screen-based ET* (as used by Muldner & Burleston, 2015) has recently gained popularity, where a device is attached to a screen and views the participant's eye from the distance (Holmkvist et al., 2011). This offers the advantages of unobtrusiveness and lower demands on the data processing (compared to mobile ET), in particular since it is straightforward to relate gaze directions to the displayed content on a screen. Additionally, screen-based ET is usually much more affordable than mobile ET. If fully fledged hard- and software are bought, the screen-based technology is currently substantially less expensive than the mobile one.

Among *head-mounted eye-trackers*, *mobile ET with goggles* (as used by Schindler et al., 2016), which are relatively lightweight (approx. 100 g), is increasingly used in educational research. It has the advantage to allow students to work on paper and pencil tasks. Especially in geometry MSTs, it may be unfamiliar for the students to solve MSTs on a computer screen, where scribblings, writings, and drawings have to be carried out differently from what many students are used to. In mobile ET, students can work on a sheet of paper, move it around, can move their heads and body relatively freely. We see that mobile ET avoids the possible influence that screen-based ET may have on students' creative processes, while is provides accuracy and precision similar to remote eye-trackers.

## Data analysis of ET in creativity research

The analysis of ET data provides a view on internal cognitive processes underlying students' actions, which can complement the insights gained from studies of products or the observation of students. We illustrate three methods of ET analysis that we consider useful for studying creativity. They apply to both screen-based and mobile ET.

*Gaze overlaid videos.* This qualitative analysis draws on gaze overlaid, augmented videos, where participants' focus is visualized by a circle wandering in the way their eye movement does. Researchers can have a detailed look at what students are paying attention to and how their focus of attention shifts. Schindler et al. (2016) found that the analysis of ET overlaid videos can contribute to revealing "how new, creative ideas evolved, to reconstructing approaches that were complex and whose written/drawn descriptions did not allow to clearly reconstruct them, and to evaluating the degree of elaboration of students' approaches" (p. 168). However, we see that analyzing gaze overlaid videos can be extraordinary time consuming; and thus suggest to combine it with other methods, such as attention maps, in order to preselect relevant episodes or approaches.

*Attention Maps (Heat Maps) and Scan Paths.* Attention maps are a representation of ET data most often visualized as *heat maps*. Heat maps show the distribution of gaze points over a certain period, projected onto the task sheet. Areas where the participant looked at often are usually colored in warm colors (red); areas where only few fixations applied, are marked in colder colors (blue) (cf. Holmkvist et al., 2011). In creativity research, heat maps can be used to get an overview on what kind of approach a student had when solving an MST: It provides researchers with an image of where the main foci of attention were. This can be used to categorize students' approaches or to sort them—for instance, for preselecting approaches for a further qualitative analysis. Similar to attention maps, *scan paths* provide information on where the participants looked at. In addition, they display the sequence of fixations by numbered, connected points (representing the fixations) that are connected through lines or arrows in the order of appearance. It is even possible to represent the fixation times through the size of the displayed points. We see that scan paths can be used to portray students' approaches; and even allow for a glimpse on the problem-solving *process* and, thus, for studying creativity in a process view.

*Statistical measures.* Data from ET devices can furthermore be quantified, which can be understood as the counterpart of the qualitative analyses mentioned above. The analysis is conducted based on the data, not on the visualization. What can be measured are, e.g., the movement directions and their velocity, the position duration, the numbers of saccades or fixations, and the saccadic latency. Muldner and Burleston (2015) conducted, e.g., statistical analyses comparing a low- and high-creativity group of participants. They found that less creative students had a significantly shorter saccade path length indicating that they "may have focused more locally when generating their proofs, since shorter saccades suggest they were looking at (…) objects in close proximity" (p. 135). Regarding its value for creativity research, this kind of analysis enables researchers to, e.g., find quantifiable differences in the eye movement data of different groups of participants solving MSTs.

## DISCUSSION AND OUTLOOK

Existing research using well-established research methods has developed a considerable body of knowledge on mathematical creativity and students' work on MSTs. We see that ET and its analysis can constitute an additional research tool in the "tool box" of researchers aiming to investigate mathematical creativity; and might reveal findings that can add to the body of knowledge. This paper gave a glimpse on the opportunities that ET may hold for investigating mathematical creativity. We think that empirical research still has to prove if our ideas and visions on how to use ET for investigating creativity can hold or have to be revised; and how they can be specified. We expect ET to get closer insights into, e.g., how students get to "aha!" moments; in particular to shed more light on the incubation phase in students' creative processes. In addition, ET can be an additional research tool for anticipating students' subsequent steps in solving MSTs; as eye movements potentially precede people's actions. Furthermore, we see that automatic analysis through pattern recognition and machine learning approaches offers various opportunities to analyze students' creativity both in a product and in a process view in the future. This can contribute to effectively and quickly analyze students' creative processes, to finally foster them adequately in their creativity, and accordingly prepare them for their future lives in modern, creative societies.

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# TEACHERS' PERCEPTION IN MEETING THE NEEDS OF MATHEMATICALLY GIFTED LEARNERS IN DIVERSE CLASS IN BOTSHABELO HIGH SCHOOLS AT MOTHEO DISTRICT

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***Abstract.*** *Diversity is one of the country's greatest assets, where teachers are faced with many challenges in their classroom. Many teachers have the willingness to cater for the needs of mathematically gifted learners, but lack the knowledge and skills to be able to do as successfully as they are not adequately prepared enough for effective teaching to meet the needs of gifted learners. The aim of this study was to examine the perception of teachers regarding mathematically gifted learners in diverse class. The study was conducted among 20 high school teacher in Botshabelo. A mixed methods was used and data was collected through questionnaire. The results of the study discovered that teachers' have positive perceptions regarding gifted learners in mathematic based on the characteristics of the gifted and the different provision provided to meet the needs of the gifted and to shed more light on the plight of gifted students in the diverse classrooms.*

*Key words: diversity, mathematically gifted learners; characteristics; learner's needs identification*.

## INTRODUCTION

In schools teachers have the willingness to cater for the needs of mathematically gifted learners to meet the diverse needs but lack the knowledge and skills to do so successfully (Oswald & De Villiers, 2013).In South Africa the needs of mathematically gifted learners are still a cause for concern as they continue to fade into the background of the regular classroom with more visible needs of struggling learners overshadowing those of the gifted learners (Marzano & Boogren, 2010). Exceptionally able students require greater extension of breadth and depth of learning activities than is normally provided for the main cohort of students.

The study investigate teachers' perceptions about the ways they address the needs of gifted by identifying characteristics of mathematically gifted learners in order to be able to provide the appropriate support and guidance to these learners.

### Identification of mathematically gifted.

In order for mathematically gifted learners to be identified, teachers need to have knowledge of the characteristics of giftedness (Banks & Banks, 2009).Numerous researchers have shown that it is the individual teacher that plays the central role in identifying and providing for the gifted and talented child ( Clark, 2005; Reis & Small, 2001). It is therefore up to the individual teacher to identify and provide the appropriate programs for gifted and talented students within their class. Identification process of gifted individuals is an important issue in order to provide support with regard to their abilities. According to (Aljughaiman & Ayoub, 2012), the teacher should know what gifted learners already know about the subject, so that pre- assessment can be given. In addition the teacher should extend the curriculum to include interesting activities, challenging problems, games or puzzles that learners will want to do independently. However the teacher should orient the learner to interact with other learners for beneficial lessons.

### Characteristics of mathematically gifted

Mathematically gifted are different in their cognitive abilities, motivation personalities, self –sufficiency, emotional control and learning style (Rotigel & Fello, 2004). Teachers need adequate knowledge about the gifted diverse characteristics, so that they can nature their individualities and will help in providing gifted students with an appropriate education that will meet specific needs of these students (Davis & Rimm, 2004)

The range of characteristics exhibited by gifted learners is inexhaustible. gifted students are different from other student in their learning in five ways:

- They learn new concept quickly
- They remember previous experience which can make reviewing what they have been studied boring for them
- They perceive concept and ideas at more complex and abstract level than their classmate
- They become frustrated by being made to shift interesting topic as they see it other learning task, before they have learned the whole topic
- They have heightened powers of concentration

### Characteristics of mathematically gifted

Reasoning skills

- Ability to abstract, conceptualize and synthesize (can see similarities, pattern and differences, can generalize from one situation to another), and find pleasure in intellectual activities.
- In the class room the gifted learners need exposure to variety of materials and concepts, opportunities to peruse multi-disciplinary topics and themes.
- Possible problems include boredom with classroom instruction, unresponsiveness to traditional teaching methods, and being judge as "lazy" and unmotivated by the teacher and considered too serious by peers.

Knowledge based

- Known a great deal about a variety of topics, has quicker mastery and recall 1 of actual information than other children of the same age.
- Instructional classroom needs include early instruction of basic skills with minimum of repetition and drill, and exposure to new and challenging information about cultural, economic, environmental and educational issues.
- Possible problems include rebellion at having to work below one's level of competence, development of poor work habits because of lack challenge peer resentment of skills and achievement.

Intense curiosity

- Questioning and inquisitive attitude, ask many unusual or provocative questions, interested in the "why" and "how" concerned with what makes things right or wrong, has areas of "passionate" interest inside or outside of the school.
- Educational needs include opportunities for active inquiry, and instruction in how to access information and conduct research. (Davis & Rimm, 2004)

## The needs of diverse mathematically gifted learners

Mathematically gifted learners have basic needs as other learners, like love, understanding, encouragement to grow, support, guidance, security, acceptance, companionship. However, they also have special needs which correspond to their special natures (Davis & Rimm, 2004). To ensure effective learning for all learners in the classroom, teachers need to develop sensitivity to individual needs and respond to them. The following are classified as special needs of diverse mathematically gifted that teachers need to take into consideration (Oswald & De Villiers, 2013),

- A flexible program which involve the higher cognitive concepts and progress as defined by Bloom and Guilford
- Freedom from restrictions of structured requirements and limited time frames
- Time and freedom to experiment, explore subjects of interest
- Opportunity to brainstorm, thus producing creative ideas.
- Encouragement to ask questions, make discoveries, pursue own interest in depth.
- Friendly recognition and acceptance of their giftedness.

To create a challenging learning environment, teachers should maximize opportunities to help gifted students learn, grow and be challenged. They should also realize that each gifted student has an individual learning profile of his/ her intellectual and affective needs, abilities, multiple intelligences and learning preferences (Chessman, 2006; Dai, 2004).Teachers need to create a caring, socially rich, and cooperative classroom where differences are accepted. The study of (Siegel, 2003) ,suggest that by providing materials effectively in school can improve achievement in mathematics and the home learning environment such as parental education have significant effect on students' performance.

## Methodology

Both qualitative and quantitative approach was used to investigate teachers' perceptions about the ways they address the needs of gifted learners, the characteristics of mathematically gifted learners and the needs of gifted in order to be able to provide the appropriate support and guidance to these students. 20 high school teachers teaching mathematics grade 11 at Botshabelo in Motheo District take part in the study. Permission was given by the principals of the schools and from the Department of Basic Education. All participants were given the questionnaire and it was collected he next day.

## Results and Discussion

| Attitude | Yes | | No | | Not Sure | | Neg/Positive |
|---|---|---|---|---|---|---|---|
| | Freg | % | Freg | % | Freg | % | |
| I believe in and support gifted learners | 12 | 60 | 5 | 25 | 3 | 15 | positive |
| I am Professionally prepared to work with gifted learners in my class | 0 | 00 | 20 | 100 | 0 | 100 | Negative |
| I am prepare to receive support from DBE to improve in teaching gifted learners | 17 | 85 | 1 | 5 | 2 | 10 | Positive |

Table 1: Perception of teachers concerning the diverse needs of mathematically gifted

Table 1 above shows that most of the participants have a positive attitude perception 12(60%) of the participant stated that they believe and support the needs of the gifted diverse learners in mathematics, using their gain experience in teaching.

Being professionally prepared all 20 (100%) participants stated that they are not professionally prepared enough to teach this learners because they never received any training regarding this learners but 17 of the participants (85%) are willing to receive support to improve in teaching mathematically gifted.

The overall analysis shows that most of the teachers have positive attitude towards teaching mathematically gifted to meet the diverse needs of learners and they are also willing to receive support from DBE to improve their skills and knowledge for effective teaching and learning as they sometimes don't know how to handle/ treat which results in having negative attitude towards them but they try their best to support them.

Participants were asked to evaluate their provision in supporting the needs of mathematically gifted learners in diverse class. The pie chart below in fig 1, shows the provision given to learners in classroom on daily basis.

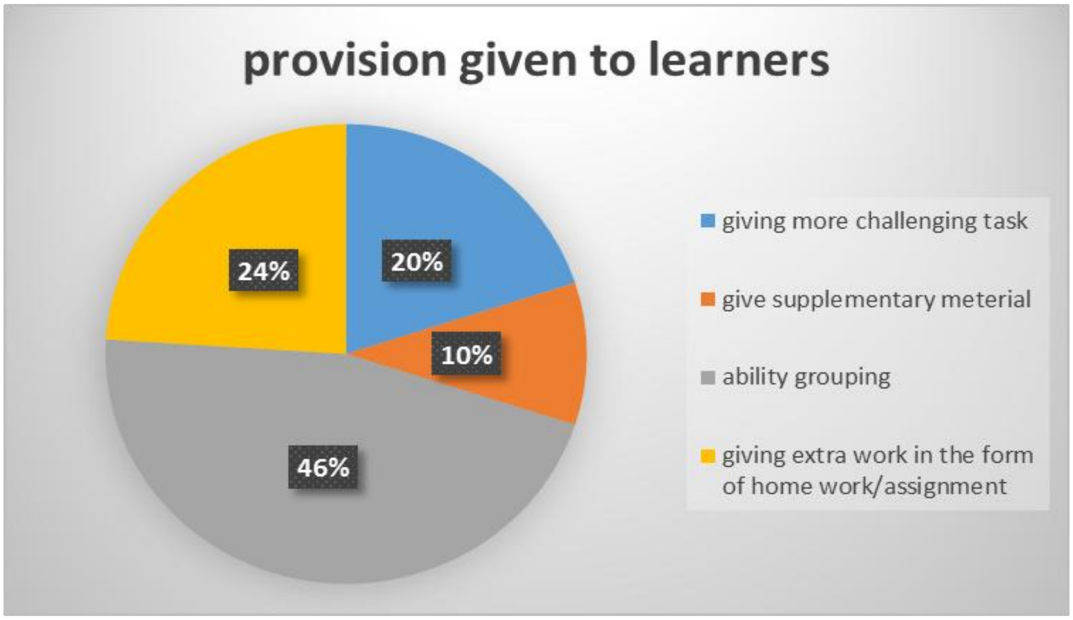


Figure 1: Provision given to mathematically gifted leaners in diverse class

Looking at the results in fig 1, it is clear that majority 46% of the participant's use ability grouping and 24% give learners extra work in the form of homework or assignment on daily basis to meet their needs. According to (Van Tassel-Baska & Brown, 2009) teachers must use differentiated instructional strategies to meet the diverse needs of gifted learners otherwise, they can become frustrated. The participants also quoted that even though they are trying to give provision in their class, they are still faced with many challenges. These challenges include lack of adequate learning materials, overcrowded classes, indiscipline cases, and teachers being unskilled.

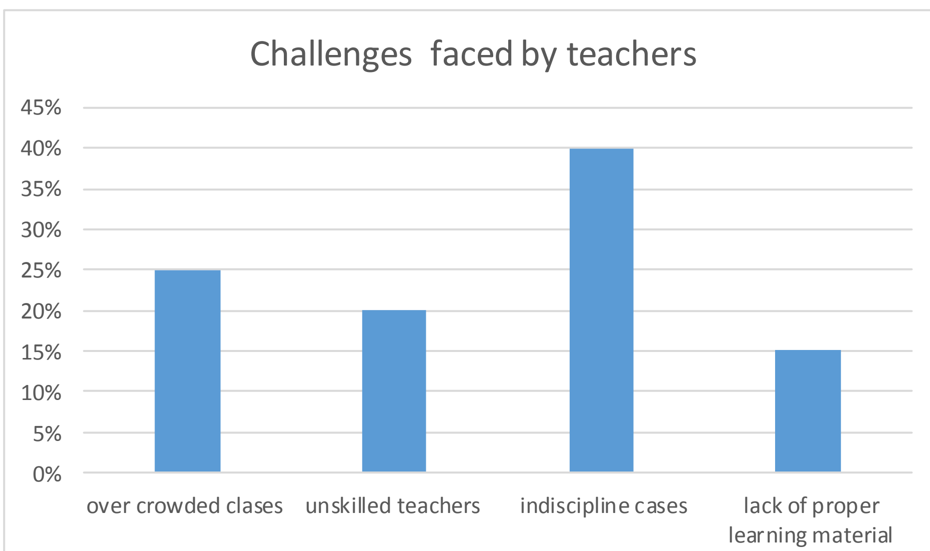


Figure 2: Challenges faced by the teachers

40 % of the participant indicated that because of overcrowded classroom, gifted learners have no discipline, often making noise and disrupting other classmate that is why they keep them busy all the time.

## Conclusion

Many mathematically gifted learners in regular classroom remain invisible, even when their special talent are acknowledged, little is done to further their development. The mission statement of the department of education is to ensure that the educational needs of all learners are met so that their potential can be fully developed, unless some provision is made that recognizes this advanced knowledge, bright kids learn nothing new until January(Davis & Rimm, 2004; Oswald & De Villiers, 2013). Exceptionally gifted learners are leaners at risk in schools. Gifted learners suffer both academic and psychosocial problems because the education provided to them does not match their unique intellectual needs(Van Tassel-Baska & Brown, 2009).It is hoped that if teachers can understand gifted learners unique cognitive characteristics, their talents can be appreciated and further developed. However Despite the development of an inclusive education policy to address this exclusion, one of the issues that hampers progress is the lack of teacher skills in adapting the curriculum to meet a range of learning needs (Dai, 2004). A possible solution lies not only in policy change in regard to gifted education, but also in training teachers to identify gifted learners and to effectively facilitate their learning in the regular classroom

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# STRATEGIES USED BY TEACHERS FOR SUPPORTING MATHEMATICALLY GIFTED LEARNERS IN BLOEMFONTEIN HIGH SCHOOLS

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***Abstract.*** *Most gifted learners in mathematics fall through the cracks of inclusive classrooms as are taught by the same methods used to teach other learners. The needs of the gifted learners are often ignored as teachers believe that they can learn on their own without special programs. Teachers' strategy is a key element for educators to provide proper learning instructions to gifted learners. The purpose of this study was to investigate strategies which teachers use when they support mathematically gifted learners in their inclusive classrooms. Nineteen teachers from high schools around Bloemfontein participated in the study. Data was collected through questionnaires.*

*Key words: Mathematically gifted leaners, teachers' strategies, inclusive classrooms*

## INTRODUCTION

Mathematically gifted learners differ from their non-mathematically gifted peers in the ways in which they learn the best. Not only do gifted learners learn at a faster pace, they also get motivated when they learn in greater depth. It seems that every teacher in every classroom would have the will and the skill to adjust the curriculum to suit the needs of the diverse learners in their classrooms. However, there is enough evidence that the current classrooms fail to meet the needs of all learners, especially the gifted ones (Callahan & Hertberg-Davis, 2012). While many teachers have long recognized the serious shortage of care in addressing the needs of mathematically gifted learners in South Africa, the Marland Commision in 1972 concluded that if Gifted and Talented learners are, in fact, deprived and can suffer psychological harm to function well which is equal to or greater than the similar deprivation suffered by any other group with special needs served by the Department of Education (Marland, 1972). This paper examined how a gifted learner is being identified including strategies to support mathematically gifted learners.

### Identification of a gifted learner.

Teachers' nomination play an important role in identifying a specific ability, such as the ability in mathematics. Teachers might nominate learners as gifted based on the extent of learner's performance (Heller, Perleth, & Lim, 2005). Eyre(2001) suggested that the 'diagnostic assessment' as one of the broad forms of information available in schools be used to overcome problems associated with different tests.

### The needs of the mathematically gifted learners.

Teachers need to look at ways that are suitable for varied levels of learning in the mathematics classrooms. For example, teacher strategies include grouping learners and providing learners with sufficient and broad challenging experiences (Marumo, 2017) which enhance the learning of all learners. Thus, differentiation is the process by which curriculum objectives, teaching methods, assessment methods, resources and learning activities are planned to cater for the needs of all individuals in ways which meet their needs. Rotigel & Fello (2004) give an example of differentiation in teaching, they explain that for example when calculating the area of polygons, the average learner is taught the basic formula: this approach requires low order thinking skill. On the other hand, the gifted learner is exposed to higher order thinking skill where calculations involve various real-world applications of the calculating area.

## MATERIALS AND METHODS

The study used qualitative-quantitative approach in examining strategies used by teachers for supporting mathematically gifted learners. A total of 19 grade 10 high school teachers from Bloemfontein, South Africa participated in the study. All the participants were mathematics teachers. Permission was obtained from the Free State Department of Basic Education as well as the principals of the schools prior to conducting the study. Questionnaires were hand delivered to the participants at 10 schools and were collected after 2 days. Moreover the participants completed the questionnaires at their schools, and in case of the absent teachers, the questionnaires were left to school administrators.

## RESULTS

The first aspect examined in the study is the methods which teachers use to identify gifted learners in the inclusive classrooms.

| Method | Yes | | No | | Not Sure | | Total | |
|---|---|---|---|---|---|---|---|---|
| | Freq | % | Freq | % | Freq | % | Freq | % |
| Nomination | 13 | 68.4 | 5 | 26.3 | 1 | 5.3 | 19 | 100 |
| Assessment results | 16 | 84.2 | 2 | 10.5 | 1 | 5.3 | 19 | 100 |
| Identification by other teacher or previous school | 2 | 10.5 | 12 | 63.2 | 5 | 26.3 | 19 | 100 |
| Identification by parents | 2 | 10.5 | 14 | 73.7 | 3 | 15.8 | 19 | 100 |
| Other methods | 2 | 10.5 | 12 | 63.2 | 5 | 10.5 | 19 | 100 |

Table 1: Methods used by the teachers to identify gifted leaners.

The results show that the most widely used method by the teachers to identify gifted learners in mathematics classrooms centered upon nomination and assessment results. About 84% of the participants cited assessment results and approximately 68% mentioned nomination as observing the leaner's performance during question and answer sessions. A smaller proportion just above 10% mentioned identification by other teacher or previous school, identification by parents and other methods

To further explore the issue of the identification of gifted learners, the present study asked the participants to evaluate the identification process of mathematically gifted learners in their classrooms?'

| Level of the identification process | Frequency | Percentage |
|---|---|---|
| Easy | 6 | 31.6 |
| Very easy | 1 | 5.3 |
| Neither easy nor difficult | 9 | 47.4 |
| Difficult | 3 | 15.8 |
| Very difficult | 0 | 0 |
| Total | 19 | 100 |

Table 2: Level of the identification process.

From table 2 it is clear that a large number of the respondents (47.4%) were not sure about how a gifted learner can be identified. A further 31.5% indicated that the process was easy, but just over 15% mentioned that it was difficult. Figure 1 below shows the grouping strategies used by teachers to group learners in their regular classrooms.

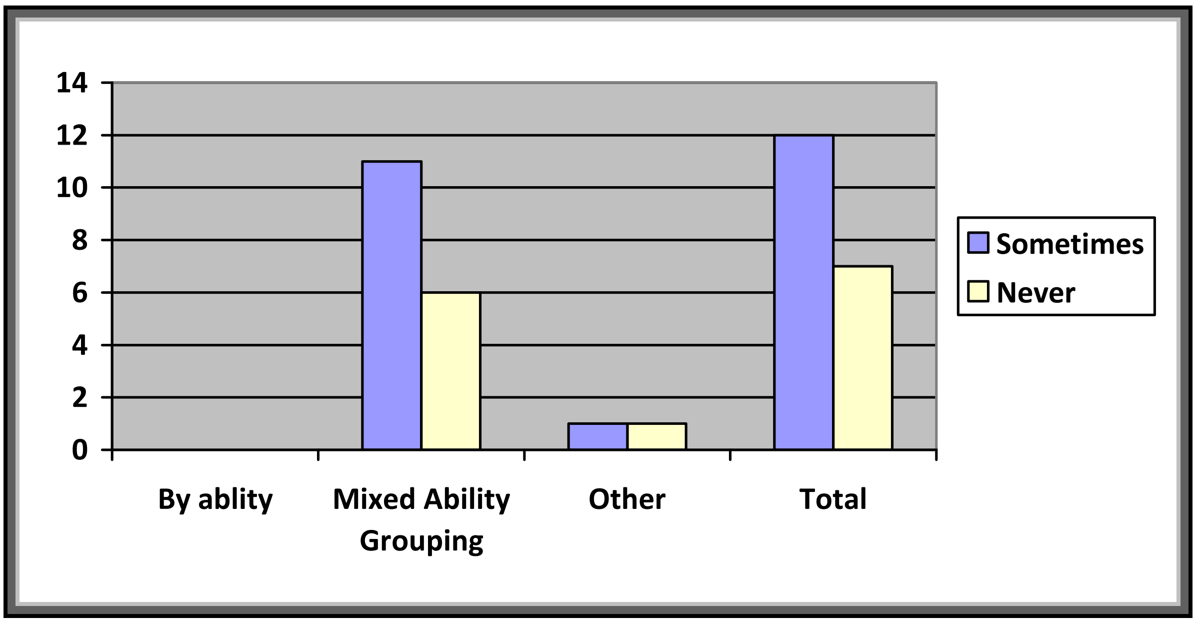


Figure 1: Grouping strategies for the learners.

The majority of the teachers (17) appear to use the mixed ability grouping in the classrooms in two ways. Most of the teachers (11) which use the mixed ability grouping indicated that they sometimes change the groupings by swapping the learners between different groups. Fewer teachers (6) in the same mixed ability grouping however prefer not to change groups. None of the teachers chose 'grouping by ability' as one of the grouping strategies.

All two teachers who cited other grouping strategies explained that they use the merit list to group the learners. Table 3 shows teachers' explanations as their reasons for re-arranging the groups in the classrooms.

| Reasons for rearranging | Frequency |
|---|---|
| Leaners get motivated when they interact with different peers | 2 |
| To help underperforming learners | 4 |
| Learners enjoy different topics, they are grouped according to the classroom activity | 3 |
| According to their performance in the given test | 2 |
| Not answered | 2 |
| Total | 13 |

Table 3: Reasons for rearranging the leaners' groups in the classrooms.

As noted above, most teachers rearrange their grouping based on learners who act as peer tutors and the classroom activity. There are few teachers who swap their learners either for leaners to get motivated when they interact with different peers or based on the test result.

**CONCLUSIONS**

The inclusive education of South Africa is offered through the lens of disability (Mhlolo, 2015). Therefore the policy does not encourage the teachers to meet the needs of gifted learners. Even though the teachers were aware that mathematically gifted learners should be given more challenging tasks than their classmates, the teachers do not know how to assess these student in the inclusive classroom settings. Therefore this study suggests that the curriculum should be revised and include strategies which are aimed at developing mathematically gifted learners in the inclusive classrooms.

**ACKNOWLEDGMENTS**

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# ANALYSIS OF A GIFTED PRIMARY SCHOOL STUDENT'S ANSWERS TO A PRE-ALGEBRA TEACHING UNIT

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***Abstract.*** *We present a teaching unit aimed to introduce students to algebra and solution of linear equations through geometric pattern problems. We analyse the answers, to the teaching unit, made by a 9-years-old mathematically gifted student in grade 5. We have classified his answers in several styles of strategies of use of graphical information and different types of generalisation. We have also analysed the student's capacity to transfer algebraic knowledge to other algebraic contexts different from geometric pattern problems. Our results show that the student exhibited several characteristics of mathematical giftedness related to his strategies of solution of problems, which are notably different from those strategies that, according to the literature, are used by average students in the same grade.*

*Key words: Geometric pattern problems, Giftedness, Pre-algebra, Primary education*

## INTRODUCTION

Algebra is a key topic in secondary mathematics education, since it is the tool to solve new kinds of problems that otherwise cannot be solved. However, most students have difficulties in learning algebra, related to the meaning of letters and the equal sign, the notions of variable and unknown, and transforming word statements into algebraic expressions (Banerjee & Subramaniam, 2012; Jupri, Drijvers & Van den Heuvel-Panhuizen, 2015), among others.

Geometric pattern problems are a context used to develop algebraic thinking in primary school (García-Reche, Callejo & Fernández, 2015; Merino, Cañadas & Molina, 2013; Rivera, 2013), a type of thinking that is based on dealing with variables and unknowns without operating symbolically with them (Radford, 2000). Some authors have reported that this context seems very useful and fruitful for mathematically gifted students in order to start learning pre-algebraic contents. Amit & Neria (2008) made a detailed analysis of the strategies used by gifted students of grades 6 and 7 to solve pattern problems aimed to generalisation. Fritzlar & Karpinski-Siebold (2012) differentiated five components of algebraic thinking by analysing the differences among the ways of thinking of grade 4 students, including gifted students.

Several manipulative models of equations are used by teachers and researchers. One of them is the *balance*, that allows to represent and manipulate the terms of an equation in a balance. This model helps students to conceptualize the solution of linear equations as a process of compensation to maintain the balance horizontal.

A research question is how design teaching units to introduce gifted students into algebra in a meaningful way. In this context, the specific objectives of the research we present are:

i) To analyse the strategies of solution and ways of generalisation used by a mathematically gifted student when solving geometric pattern problems.

ii) To analyse some aspects of the learning trajectory of a mathematically gifted student while working on a pre-algebra teaching unit.

## THEORETICAL FRAMEWORK

Geometric pattern problems show (Figure 1) a graphical representation of the first terms of an increasing sequence of natural numbers, and pose students some questions about the terms of the sequence. These problems include different kinds of questions. The most common *direct questions* are to calculate the values (i.e., the number of pieces in the graphical representation) of *immediate*, *near* and *far terms* of the sequence (Stacey, 1989) and to *express a general rule* to calculate any term of the sequence. The *inverse questions* give the value of a term and ask for the position of that term in the sequence (Rivera, 2013).

A characteristic of geometric pattern problems is the graphical information in the statements. Students use it in different ways. Rivera & Becker (2005) distinguished between *figural* and *numerical* strategies. Figural strategies are those that derive generalisation from analysing and decomposing the graphical representation of the terms. Numerical strategies are those that use the numerical values of the terms, without paying attention to the graphical organization of the pieces.

Students also follow different strategies to calculate the numerical value of terms in the direct questions. García-Reche, Callejo & Fernández (2015) described the strategies *counting*, *recursive*, *functional* and *proportional*. The counting strategy consists of drawing the graphic of the term and counting its pieces. The recursive strategy is based on adding the numerical difference among consecutive terms to calculate the next term. The functional strategy uses the position of the term to calculate the number of pieces asked. The proportional strategy uses a supposed relationship of proportionality between the positions of two terms and their values to calculate the value of one of the terms.

A key component of geometric pattern problems is generalisation. Radford (2006) distinguished five levels of generalisation: *naïve induction* (trial and error guessing), *arithmetic* (recursive calculations), *factual algebraic* (a general rule expressed only by particular numbers), *contextual algebraic* (a general rule referring to any term, but expressed verbally and contextualized), and *symbolic algebraic* (a general rule expressed by using alphanumeric symbols).

Küchemann (1981) classified the meanings students assign to letters in algebraic expressions and the ways they use the letters into six different types: *evaluated*, *not used*, *object*, *specific unknown*, *generalised number* and *variable*.

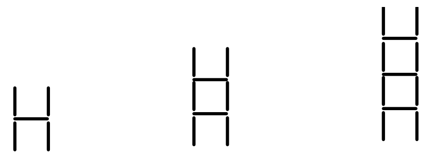
Mathematically talented students have some abilities especially useful to solve geometric pattern problems, like *locating the key of problems*, *identifying patterns and relationships*, *generalising* and *transferring ideas from a context to another*, *developing efficient strategies*, *abbreviating solution process* and *quickness of learning* (Freiman, 2006; Greenes, 1981).

## RESEARCH METHODOLOGY

We present results from a case study based on a 9-years-old gifted student who followed an experimental teaching unit for pre-algebra based on geometric pattern problems. The student had been identified as gifted according to the usual criteria (IQ≥120). Furthermore, we had personal information about his problem solving behaviour in other areas of mathematics because he had attended a workshop for gifted students. The experiment was completed when the student was starting grade 5 of primary school. The sessions were conducted by the first author by videoconference (Skype) and were video-recorded.

We designed and implemented a teaching unit divided into three parts. The first part was aimed to teach the student to generalise functional relationships from geometric patterns and solve inverse questions. It consisted of 20 problems, with 15 linear functions (Figure 1) and 5 quadratic functions (Figure 2), all of them including three direct questions ($a$ to $c$) and an inverse question ($d$).

We have to calculate the number of pieces of wood necessary to make a ladder with many steps.



1 Step 2 Steps 3 Steps

a) How many pieces are needed for a ladder with 4 steps? How do you know it?

b) How many pieces are needed for a ladder with 10 steps? How do you know it?

c) How many pieces are needed for a ladder with 45 steps? How do you know it?

d) If we need 26 pieces to make a ladder, how many steps does the ladder have?

Figure 1: A linear geometric pattern problem used in the first part of the teaching unit.

A snake grows in this way:

Day 1 Day 2 Day 3

a) How many triangles shall the snake have in day 5? How do you know it?

b) How many triangles shall the snake have in day 10? How do you know it?

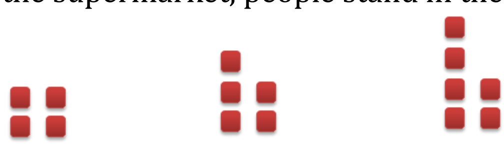
c) How many triangles shall the snake have in day 40? How do you know it?

d) If the snake has 37 triangles, which day will it be? How do you know it?

Figure 2: A quadratic geometric pattern problem used in the first part of the teaching unit.

The second part was aimed to teach basic algebraic concepts, like the meaning of letters, translation of verbal expressions into algebraic ones and solution of linear equations. It consisted of 6 problems (Figure 3), all of them including a direct question ($a$), an algebraic generalisation question ($b$) and an inverse question ($c$).

Waiting in the queue at the supermarket, people stand in the following way:



Minute 1 Minute 2 Minute 3

a) How many people will be waiting in minute 12? How do you know it?

b) Write down the formula you have used in the previous question.

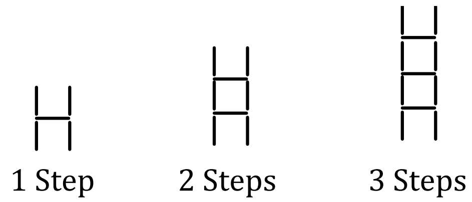
c) If there are 49 people waiting, how many minutes have passed?

Figure 3: A geometric pattern problem used in the second part of the teaching unit.

In this part, like other authors (for instance, Store, Richardson & Carter, 2016), letters were introduced as objects by using the first letter of the unknown. In addition, we used a virtual balance to introduce the student into linear equations.

The third part of the teaching unit focused on reinforcing the algebraic contents learned. We selected 7 problems from the first part, in which the student was only able to solve the inverse questions by trial and error, and we modified their statement asking (Figure 4) for an algebraic generalisation (*a*), to simplify this algebraic expression (*b*), and an inverse question (*c*). We also posed 6 algebraic word problems (Figure 5) and 4 linear equations to practice the routine of solving equations.

We have to calculate the number of pieces of wood necessary to make a ladder with many steps.




a) Write down an algebraic formula to calculate the pieces of wood necessary for any number of steps.

b) Do you believe that it is possible to get a simpler formula? If so, write it down.

c) If we need 41 pieces to make a ladder, how many steps does the ladder have?

Figure 4: A variation of the problem in Figure 1 used in the third part of the teaching unit.

Hector and Mary are counting how many color pencils has each one. Hector has twice as many pencils as Mary plus 7 pencils. If Hector has 33 color pencils, how many does Mary have?

Figure 5: An algebraic word problem used in the third part of the teaching unit.

## ANALYSIS OF STUDENT'S ANSWERS

In the first part of the teaching unit, the student generally used visual and functional strategies to solve direct questions. Moreover, he did mainly factual algebraic generalisations and also contextual generalisations. Table 1 shows the number of times the student used each type of strategy to answer direct questions (he did not answer questions *b* and *c* in one problem).

| | Use of graphical information | | Calculations for direct questions | | Ways of generalisation | | |
|---|---|---|---|---|---|---|---|
| | Visual | Numerical | Recursive | Functional | Arithmetic | Factual | Contextualized |
| *a* | 12 | 8 | 5 | 15 | 5 | 8 | 7 |
| *b* | 14 | 5 | 1 | 18 | 1 | 12 | 6 |
| *c* | 14 | 5 | 0 | 19 | 0 | 12 | 7 |

Table 1: Strategies used to answer direct questions in the first part of the teaching unit.

To solve the inverse questions in the first part of the teaching unit, the student used different strategies depending on the type of generalised expression he produced in the direct questions (Table 2). If the generalisation was of type $y=ax\pm b$, the student was able to correctly get the answer by reversing the calculations, with only two exception. In contrast, if the generalisation was of types $y=ax+b(x\pm c)\pm d$, $y=x^2$ or $y=(x\pm a)(x\pm b)$, he was not able to calculate the answers because he still did not know how to solve equations, and he used trial-and-error direct calculations.

In the second part of the teaching unit, the student understood rapidly the meaning of letters in algebraic expressions, making use of the letters in a dual way: as an object (e.g., using *m* to express *minute*) and as a generalised number (he knew that letters represent a variety of values). He translated his verbal expressions into algebraic ones using the standard hierarchy of arithmetic operations and parenthesis. He meaningfully interiorized the balance model so that he was able to simulate it in the word processor. For that reason, he learned to solve equations by compensation (operating the same way in both sides) and made a great progress towards the algebraic syntax, being able to solve equations with a quasi-unnoticeable use of the balance.

| | Trial and error | Correct inversion | Wrong inversion |
|---|---|---|---|
| $y = ax \pm b$ | 1 | 6 | 1 |
| $y = ax \pm b(x \pm c) \pm d$ | 7 | - - | - - |
| $y = x^2$ | 2 | - - | - - |
| $y = (x \pm a)(x \pm b)$ | 3 | - - | - - |

Table 2: Types of generalisation and strategies for the inverse questions in the first part.

In the third part of the teaching unit, thanks to the algebraic contents taught in the second part, the student only used functional strategies, symbolic algebraic generalisations and letters as variables for the direct questions. He was also able to simplify his own algebraic expressions. For the inverse questions, he ever solved the equations in an algebraic syntactic way and he solved correctly all the algebraic word problems with little difficulty.

## CONCLUSIONS

We have presented a synthesis of the development of a research experiment aimed to introduce a 9-year-old gifted student into algebra. The student learned to use algebra understanding the meaning of letters (as variables or unknowns), translating verbal expressions of the generalised geometric patterns into algebraic expressions, giving meaning to linear equations as a balance process and overcoming the difficulties showed in the first part of the unit.

The analysis of the student's answers, taking all the geometric pattern problems solved into consideration, shows that he used mainly visual functional strategies. This demonstrate a behaviour clearly different from that of average students in the same grade since, according to García-Reche et al. (2015) and Merino et al. (2013), average students in grade 5 solve direct questions mainly by using the counting strategy, even when they are asked to calculate a far term, and only a few of them try to solve inverse questions, ever by using trial and error strategies (Rivera, 2013). Furthermore, the student showed some characteristics of mathematically gifted students, like identification of patterns and relationships, generalization, development of efficient strategies to generalize the patterns, abbreviation of solution processes and quickness of learning.

Along the teaching experiment, in the resolution of the problems, the student showed often characteristic features of mathematically gifted students, mainly abilities to identify patterns and relationships, to generalise them, to develop efficient strategies, to locate the key of problems, to abbreviate solution processes, to transfer ideas from one context to another and, particularly, a high speed of learning.

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# RE-EXAMINING THE EDUCATIONAL PROVISION FOR MATHEMATICALLY GIFTED STUDENTS ACROSS DIFFERENT POVERTY STRATA IN SOUTH AFRICA.

A Research Report

By

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*Research shows that when provisions are denied to the gifted on the basis that they are elitist, it is the poor who suffer the most because parents from affluent families have options which poorer families cannot afford. In South Africa not much research has been done to understand the extent to which the 'elitist' justification for the dismantling of gifted education has achieved parity and fairness for the poor. This paper analyses the extent to which education provision for gifted students is fairly distributed across the poverty strata. A Qualitative Secondary Analysis (QSA) research design was employed to analyse data in Pan African Mathematics Olympiads (PAMO) documents from 2000 – 2016 together with reports from a Study of the Southern and Eastern African Consortium for Monitoring Educational Quality (SACMEQ). Data were categorised according to the different poverty categories (quintiles) in order to determine whether or not provision for gifted education was fairly distributed. Results show that participants were mainly from quintile five schools with better resources and good teachers while students from poor schools are being taught by teachers who have weak pedagogical knowledge. It can be argued that those who recommended the abolishment of classes for the gifted were in fact penalizing the gifted poor because, as the results confirm, the rich can afford private education. Public school programs for gifted children would allow children who are economically disadvantaged the only opportunities they might have to develop their talents.*

*Key words: inclusive education, gifted students, elitism, quintiles,*

## INTRODUCTION

Although significant developments in gifted education were evident prior 1994, post-colonial South Africa viewed gifted education as a white elitist practice and closed down centres for the gifted with the belief that inclusive education would ensure parity and fairness. Contrary to this assumption the current state of gifted education is perceived as dismal. Research shows that the more egalitarian gifted programs attempt to be, the less defensible they are because it is the poor who suffer the most when provisions are denied to the gifted on the basis that they are elitist. Empirical evidence shows that a much greater number of gifted children come from the lower classes, because the poor far outnumber the rich and when state provision for the gifted is poor, parents from affluent families have options which poorer families cannot afford. The end result is that what was construed as fair and inclusive practices become the very source of inequity. In South Africa not much research has been done to understand the extent to which the 'elitist' justification for the dismantling of gifted education has achieved its objectives. This paper analyses the extent to which education provision for gifted students is fairly distributed across the poverty strata. The paper works within the framework of Gagné's (2015) Comprehensive Model of Talent Development. An underlying principle of Gagné's view is that while high ability (talent) has some genetic basis (giftedness), learning, practice, and environmental factors are necessary for the emergence and development of such talent. He further demonstrated that cognitive aptitudes are fairly distributed across all socio-economic groups and if the learning, practice and environmental factors are poor then the gifted students from poor socio-economic groups suffer most because the rich have other options.

## MATERIALS AND METHODS

The A Qualitative Secondary Analysis (QSA) research design was employed to analyse data in Pan African Mathematics Olympiads (PAMO) documents from 2000 – 2016 together with reports from a Study of the Southern and Eastern African Consortium for Monitoring Educational Quality (SACMEQ). The data collection procedure involved searching on SACMEQ and PAMO websites where soft copies of records (document analysis) on South Africa's participation are available. Data were categorised according to the different poverty categories (quintiles) in order to determine whether or not provision for gifted education was fairly distributed. The five quintiles were defined according to total expenditure per household as follows:

| Quintile | Total expenditure per household in rand |
|---|---|
| 1 | 0- 14564 |
| 2 | 14565-23278 |
| 3 | 23279 - 36755 |
| 4 | 36756-79152 |
| 5 | 79153 and more |

Table 1: South African School Quintiles

Data from PAMO and SACMEQ were analysed according to these quintiles in order to determine simultaneously the distribution of PAMO participants and the quality of teaching across these quintiles. Descriptive statistics were used mainly to foreground patterns that emerged in the data.

## RESULTS

The first research question aimed at understanding how South Africa preformed over the 17 years that it has participated in PAMO. Although South Africa has participated in many other international competitions, the target populations, objectives and selection for participation is these competitions would render them less ideal for understanding gifted education. For example Trends in Mathematics and Science Studies (TIMSS) targets all students enrolled at 4$^{th}$ Grade and all students enrolled at 8$^{th}$ Grade while Progress in International Reading and Literacy Study (PIRLS) targets all Grade 4 learners in the participating countries. Both programs employ rigorous school and classroom sampling techniques to ensure that important variables in the target population are present and correct in the sampled classes. As opposed to these international competitions, PAMO's objectives are specific to gifted education in that it aims to detect young talent in mathematics, to encourage and challenge mathematically gifted young people in Africa and to exchange information on curricula and teaching methods in mathematics across the African continent. Selection for participation is based on performance on some selection tests as opposed to mere attendance in a certain grade. It is for these reasons that participation at PAMO was considered more relevant to our understanding of how provisions were made for gifted students.

*Figure 1 South Africa's Performance on PAMO 2000-2016 (n=15)*

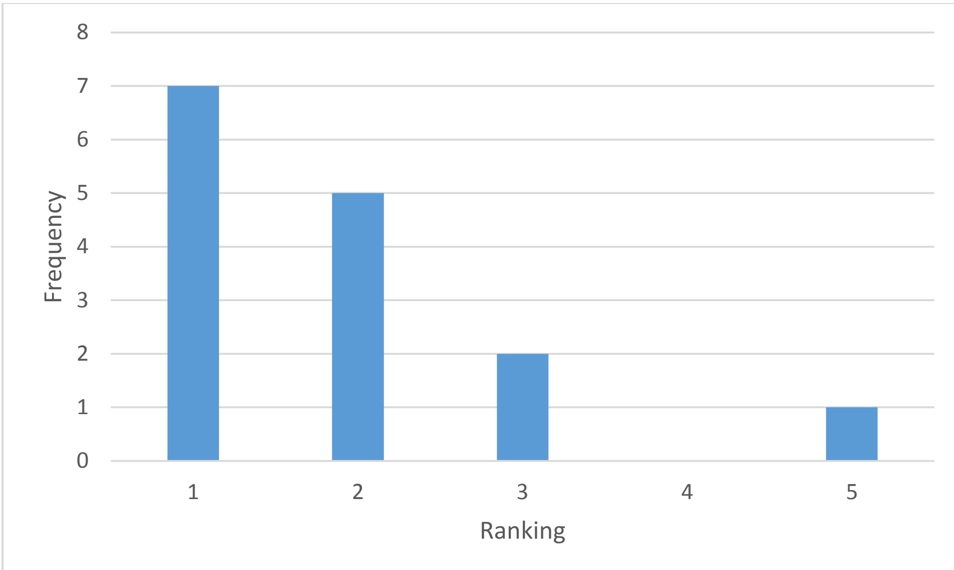


Figure 1 shows the rankings of the South African teams in the 15 times that the competition took place. The country teams were ranked first in 47% and second in 33% of their participations suggesting that the country is doing very well on these competitions. These results are in stuck contrast to how South Africa performed on other international competitions. For example South Africa participated in the Trends in International Mathematics and Science Study (TIMSS) in 1995, 1999, 2002 and 2011 and in all cases performed at the bottom of the middle-income countries. In the 2006 Progress in International Reading and Literacy Study (PIRLS), South African Grade Five pupils achieved the lowest average score when compared with all the 45 countries that participated, including other middle-income countries.

The second research question aimed at understanding how PAMO participants were distributed over the quintiles. The results show that more than 90 per cent of the participants were from quintile 5 schools and none came from quintiles one, two and three. These results are similar to observations made by the Human Sciences Research Council (HSRC) who analysed TIMSS 2011 results according to quintiles. Although South Africa's average score for mathematics was 352 the average score for quintile 5 schools was 438 while that for quintile 1 schools was 316. These results suggest that quintile 5 schools achieved a much higher average score than the other quintiles.

The third research question aimed at understanding the quality of teaching received by students across these different quintiles. There is a broad consensus that 'teacher quality' is the single most important school variable influencing pupil achievement. In order to answer this question on teacher quality SACMEQ III results were analysed. In addition to testing Grade Six pupils, SACMEQ III (2007) also tested Grade Six teachers. Using Rasch analysis SACMEQ created a teacher test score on the same scale as the pupil test score such that direct comparisons could be made between pupil and teacher content knowledge.

| | Median number of items correct on Grade 6 maths-teacher test (max 42) | Median percentage correct for full Grade 6 maths-teacher test (42 items) | Median number of items correct on Grade 6 maths-teacher test but only for 16 items common with pupil test (max 16) | % correct for common-items of Grade 6 maths-teacher test (16 items) | Number of Grade 6 maths teachers in the SACMEQ sample who wrote the maths test |
|---|---|---|---|---|---|
| Quintile 1 | 15 | 37% | 7 | 42% | 75 |
| Quintile 2 | 18 | 43% | 8 | 50% | 78 |
| Quintile 3 | 17 | 40% | 8 | 50% | 80 |
| Quintile 4 | 21 | 49% | 11 | 67% | 79 |
| Quintile 5 | 30 | 71% | 12 | 75% | 89 |
| Urban | 22 | 52% | 11 | 67% | 234 |
| Rural | 17 | 40% | 7 | 42% | 167 |
| **South Africa** | 19 | 46% | 9 | 58% | **401** |

**Figure 1 Number** ***of correct items on the SACMEQ III (2007) mathematics-teacher test for South Africa after correcting for guessing (Spaull, 2013:261)***

The results for South Africa were analysed by Spaull (2013) according to quintiles. Table 2 shows that quintile four and five maths teachers in South Africa perform much better than quintiles one, two and three teachers. Spaull went further to compare these teachers with other countries and concluded that teachers from quintile five performed at the average level of teachers from the best performing countries in the sample, while quintiles one, two and three maths teachers in South Africa perform at the average level of the worst performing countries in the sample.

## CONCLUSION

This study was premised on the view that giftedness is distributed across all socio-economic levels hence it was hoped that there would be an even distribution of participants across the five quintiles. The results however show that participants were mainly from quintile five schools – schools for the wealthy families. This exclusion is further compounded by teachers who lack sufficient pedagogical knowledge to effectively cater for the needs of gifted students. Going back to the question of whether South Africa achieved parity and fairness by dismantling schools for gifted; it can be argued that those who recommended the abolishment of classes for the gifted were in fact penalizing the gifted poor because, as the results confirm, the rich can afford private education. The ramifications of this form of exclusion are not trivial given that throughout the world there are more poor gifted children than rich ones. In South Africa more recent recommendations are that each province should have at least one public school for the gifted students. This comes from the realisation that far from being elitist, public school programs for gifted children would allow children who are economically disadvantaged the only opportunities they might have to develop their talents.

## ACKNOWLEDGEMENTS

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# HIGHER ORDER THINKING IN MATHEMATICS: A COMPLEX CONSTRUCT

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***Abstract.*** *The importance of developing Higher Order Thinking in Mathematics has been an issue of discussion for more than thirty years. However, so far there is no common formulation of higher order thinking in mathematics. During the 1980's the Iowa Department of Education (1989) suggested the Integrated Thinking Model according to which higher order thinking is a combination of Basic, Critical, Creative and Complex Thinking processes. This article aims to address this issue by investigating whether such a formulation can be endorsed for mathematics and whether it can be empirically validated with 691 primary school students using structural equation modeling. The analysis provides support, to a large extent, to the assumptions of this model.*

*Key words: Higher order thinking, Basic knowledge, Critical thinking, Creative thinking, Complex thinking processes*

## INTRODUCTION

Higher order thinking (HOT) has been interpreted differently in various scientific areas and these interpretations were often rooted in different ideas. In the field of mathematics education, researchers have not yet offered a generally accepted definition of what HOT is, how it can be effectively assessed and which its constituent components are (Haladyna, 1997).

In previous studies, we formulated what mathematical HOT is, based on the components suggested by the Integrated Thinking Model (Iowa Department of Education, 1989) and presented mathematical tasks which can be used for its assessment (Pitta-Pantazi, Sophocleous & Christou, 2013; Sophocleous & Pitta-Pantazi, 2015). According to the Integrated Thinking Model (Iowa Department of Education, 1989), the combination of basic knowledge, critical thinking, creative thinking and complex thinking processes are necessary for an individual to achieve HOT. These four components are considered interrelated and dependent on each other. The Integrated Thinking Model can be considered as a top-down analysis and an adult view of HOT. Hence, it is not clear whether it describes students' components of HOT and more specifically mathematical HOT.

Motivated by this argument and using this model as a starting point, the present study aims to examine students' mathematical HOT and its potential links and interrelationships with Basic/content, Creative, Critical and Complex Thinking processes. Even though we acknowledge the limitations of such an endeavour, the examination of students' mathematical HOT will contribute in developing comprehensive understanding of its constituting components, support the development of a theoretical background and the enhancement of mathematics instruction.

In previous studies, we suggested what may be the nature of HOT in mathematics and presented possible tasks for its assessment (Pitta-Pantazi, et al., 2013; Sophocleous & Pitta-Pantazi, 2015). The present study draws on the Integrated Thinking Model (Iowa Department of Education, 1989) and adapts tasks partially used in previous studies aiming to investigate students' mathematical HOT. In particular, the study, focuses on the following two research questions:

(a) Are there differences in students' performance in Content/basic, Critical, Creative and Complex Thinking tasks?

(b) To what extent does students' performance on mathematical HOT tasks reflect the structure of the theoretical model delineated above, and especially the associations between Content/basic, Creative, Critical and Complex Thinking?

## THEORETICAL BACKGROUND

In the late 1980's the Iowa Department of Education (1989) presented a broad view of HOT. According to this model, a combination of basic, creative, critical and complex thinking skills are necessary for an individual to achieve HOT. These four components appear to be interrelated and depend on each other. This is why the model is named "integrated".

Students' Content/basic Thinking processes refer to both accepted knowledge and metacognition. Accepted knowledge may be declarative as well as procedural knowledge and skills that individuals can immediately retrieve from what they have learned (Jonassen, 2000). Overall, the Content/basic thinking refers to the knowledge that someone needs in order to think critical, creative and complex, and at the same time, to the skills that someone uses to learn and think effectively. It is important to mention that the Content/basic knowledge does only rely on recalling accepted knowledge but also metacognition. Content/basic knowledge is knowledge which was reorganized or generated at a previous lower level (Iowa Department of Education, 1989).

The Critical Thinking process is the ability to reorganize knowledge by applying processes such as analysis, connection and evaluation of existing knowledge (Iowa Department of Education, 1989). Critical thinking includes skills such as recognition of patterns, classification of objects and labelling of their common attributes, identification of the main idea, completion of sequences, identification of similarities and differences between different objects, application of rules of logic, drawing conclusions from data, identification of causal relationships, assessment of information, identification of errors in reasoning and evaluation of ideas.

The Creative Thinking process goes beyond what is already known, the reorganization of existing knowledge and the generation of new knowledge. Creative thinking includes processes such as analogical thinking, summarizing, hypothesizing, planning, imagining, synthesizing and elaborating (Iowa Department of Education, 1989).

Finally, Complex Thinking processes involve the combination of these three components to accomplish a goal. Therefore, they combine basic, critical and creative thinking and go beyond existing knowledge and its reorganization to generate new knowledge. Complex Thinking processes include problem solving skills, designing and decision making. These processes are considered the essential core of HOT (Iowa Department of Education, 1989).

## METHODOLOGY

### Participants and Test

The data presented in this study are part of a larger study and this is a preliminary analysis. In the present study, data were collected from 691 sixth grade students, 10 to 11 years old, from 33 primary schools in Cyprus in rural and urban areas.

All students completed a test constituted by four parts, one on each of the four components of the Iowa Model (Iowa Department of Education, 1989). The components were modified to address mathematical thinking processes: mathematical Content/basic Thinking processes (5 tasks), mathematical Critical Thinking processes (5 tasks), mathematical Creative Thinking processes (4 tasks) and mathematical Complex Thinking processes (5 tasks). The test included tasks from numbers and algebra, measure and shape and statistics. In addition, verbal and visual information were included to accommodate students of different cognitive styles. Students completed the test in three sessions with duration of 40 minutes each. The test was completed in three different meetings in order to avoid students' fatigue. The three sessions took place within a period of three weeks.

### Data analysis

The answers to all of the questions apart from the creativity tasks were marked from 0-4, with 0 points assigned to completely wrong answers and 4 points to completely correct solutions. The creativity tasks were also assessed based on the three dimensions of fluency (number of correct solutions), flexibility (number of different solutions) and originality (unique or rare solutions). These three dimensions have been extensively used by mathematics education researchers investigating mathematical creativity (Leikin, 2009).

To address the second research question we used Structural Equation Modeling (SEM) techniques. The analysis was carried out using the MPLUS 7.0 software (Muthén & Muthén, 2007). More than one fit indices were used to evaluate the extent to which the data fit the theoretical model under investigation. Specifically, the fit indices and their optimal values were: (a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96, (b) the Comparative Fit Index (CFI), the values of which should be equal to or lager than 0.95, and (c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.05 (Geiser, 2013).

## RESULTS

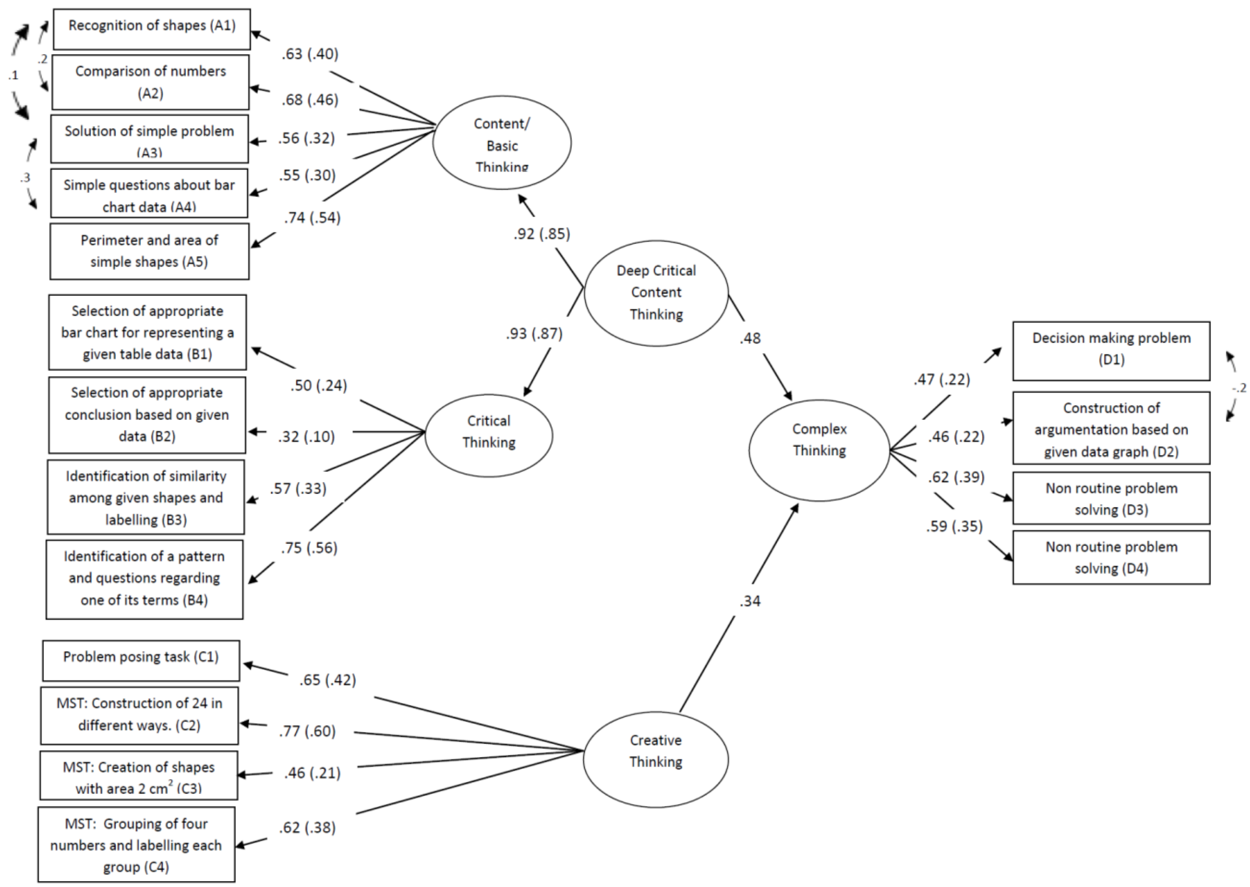
The first step to answer the two research questions was to examine whether each of the 19 tasks-variables used in the test significantly correlated with the factor to which they were designed to load. As shown in Figure 1, this was true for 17 out of the 19 tasks. Each of the 17 tasks adequately loaded on one of the four first order factors: (a) Content/basic, (b) Critical, (c) Creative and (d) Complex Thinking. The remaining two tasks were not included because they did not adequately load on any one of the four factors. The reliability coefficient of these 17 test items was Cronbach's Alpha=.81, which is considered very good.

To answer the first research question, we used the set of tasks that loaded uniquely to each of the four factors and calculated students' performance on each of the four types of thinking processes. The results showed that students were more competent in the

Content/basic tasks ($\overline{X}$=0.78, SD=0.15), less competent in the Critical ($\overline{X}$=0.66, SD=0.19), then the Creative tasks ($\overline{X}$=0.52, SD=0.17) and least competent in the Complex Thinking tasks ($\overline{X}$=0.34, SD=0.18). The use of the t-criterion for paired samples revealed that all the differences between each pair of types of thinking were significant at level $p<0.01$.

To examine the possible association within the different types of thinking processes, namely, Content/basic, Critical, Creative and Complex, we first fit a model representing the relationships exactly as described by the Integrated Thinking Model. The goodness of fit of this model, as suggested by all the three indices were not satisfactory CFI=.95, TLI=.93, $x^2$=280.13, df=113, $x^2$/df=2.48, p<.01, RMSEA=.046. Therefore, subsequent model tests were carried out revealing that the model could be improved by modifying the model in ways that agreed both with the ideas presented in the literature review as well as the tasks included in the test.

In particular, we found that all the tasks correlated with the factors which they were expected to load and all factor loadings were statistically significant, ranging from .32 to .75 (see Figure 1). Tasks (A1-A5), Recognition of two-dimensional shapes (A1), Comparison of numbers (A2), Simple word problem (A3), Simple questions about a graph chart (A4) and Calculation of perimeter and area (A5) were found to load only to one first order latent factor, that of Content/basic Thinking. In these tasks, students were asked to recall accepted knowledge and procedures, but tasks were not restricted to this. Students were also requested to apply metacognitive skills, such as "control of knowledge and task", "setting goals, correcting, controlling for errors". Tasks (B1-B4), Selection of appropriate bar chart representing given data (B1), Selection of appropriate conclusion based on given data (B2), Identification of similarity among given data and labelling (B3) and Identification and investigation of a pattern (B4) were found to load only on one first order latent factor, that of Critical thinking. The thinking skills that were required for these critical tasks were: analysis of the situation, recognition of patterns, identification of main idea, connections and comparisons of situations, logical thinking, identification of relationships and evaluation. Tasks (C1-C4), Problem posing (C1), Construction of 24 in different ways (C2), Creation of shapes with area 2 $cm^2$ (C3) and Grouping of four numbers and labelling of the group (C4), loaded only on one first order factor, that of Creative thinking. These creative tasks required students to synthesise information through analogical reasoning, hypotheses and planning, to imagine and predict and finally to elaborate by expanding, modifying, extending or shifting categories. Tasks (D1-D4), Decision making problem solving (D1), Construction of argumentation based on given data graph (D2), Non-routine problem 1(D3) and Non-routine problem 2 (D4), were found to load only on one first order factor, that of Complex Thinking. The complex thinking tasks combined the skills and knowledge required from the previous three categories of thinking: Content/basic, Critical and Creative thinking processes. For these tasks, students needed to clarify the goal of the activity in their mind, integrate, accept, reorganize and generate knowledge. Students engaged with non-routine problems whose solutions were not predictable or well-rehearsed and the pathway to their solution was not explicitly suggested. Students had to work on the problem, to make sense of it, to identify what their goal was, to design possible routes for its solution, to reach a solution, to make sense of it and finally to evaluate it. Four correlations between variables of the same factor were added to the model. This modification did not violate the theoretical assumptions of the model, since all correlations were within the same factor and not between factors.




Note. The first number indicates factor loading and the number in the parenthesis indicates the corresponding interpreted dispersion ($r^2$ )

Figure 1: Model of higher order thinking in mathematics

In addition to this, the two first order factors Content/basic and Critical thinking loaded on a second order factor which we named Deep Critical Content Thinking. This suggests that these two factors integrate. We believe that this occurs because metacognition which is a central component of basic knowledge, as described by the Integrated Thinking Model, carries connotations with Critical Thinking and this is why we believe Content/basic knowledge and Critical Thinking become integrated in a second order factor that of Deep Critical Content Thinking. It can be argued that deep conceptual understanding is part of Deep Critical Content Thinking but not restricted to this.

Finally, the factors Creative Thinking and Deep Critical Content Thinking predict Complex Thinking processes which are at the heart of mathematical HOT with an effect size of .48 and .34 respectively. The model that emerged is presented in Figure 1 and this was the model that best fitted the data (CFI=.97, TLI=.96, $x^2$=212.412, df=110, $x^2$/df=1.93, p<.01, RMSEA=.037).

## DISCUSSION

We organised our discussion based on the two research questions. Regarding the first research question "Are there differences in students' performance in Content/basic, Critical, Creative and Complex thinking tasks?", the results show that students were more successful in the tasks that required content/basic thinking processes, less successful in the critical, and creative thinking tasks in descending order and least successful in the complex thinking tasks. This may not come as a surprise, since most often Content/basic knowledge is the one that most curricula emphasize, including the Cypriot curriculum. Then, Critical Thinking processes may be addressed in a larger extent than Creative Thinking processes, but those that are the least addressed in primary school classrooms seem to be the Complex Thinking processes. Furthermore, one may argue that is not only the emphasis of the curricula that are revealed in students' scores but also students' abilities and mastery of these three different thinking processes.

Regarding the second research question "To what extent does students' performance on mathematical HOT tasks reflect the structure of Integrated Thinking Model, and especially the associations between Content/basic, Creative, Critical and Complex Thinking?", the data suggest that mathematical HOT is not a single construct, but a set of four integrated constructs: (a) Content/basic, (b) Critical (c) Creative and (d) Complex thinking. It is interesting that Content/basic thinking processes and Critical thinking processes although necessary, are not sufficient stepping stones to Complex Thinking processes. The Content/basic Thinking and Critical Thinking although distinct, they create a higher order latent construct, that of Deep Critical Content Thinking. We conjecture that this happens since metacognition which is an essential part of Content/basic knowledge carries connotations with Critical Thinking processes. The findings of this study suggest that Deep Critical Content Thinking is a strong predictive factor of Complex Thinking. In addition to this, Creative Thinking is also a strong predictive factor of Complex Thinking processes. Therefore, the findings lend support to the Integrated Thiking Model (Iowa Department of Education, 1989), which suggests that the four constructs are necessary for an idividual to reach mathematical HOT. Although the Integrated Thinking Model was designed to offer guidance to curricula development and to the design of teaching instruction, the present study confirmed that primary school students' HOT in mathematics may evolve based on their Content/basic, Critical, Creative and Complex thinking processes.

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# WHAT IS THE RELATIONSHIP BETWEEN CRITICAL THINKING AND PROBLEM POSING ABILITY?

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***Abstract.*** The aim of this study is to investigate the relationship between students' mathematical problem-posing ability and their critical thinking in mathematics. The necessity to conduct this study arises because of the limited research findings on this relationship in mathematics education research and the conflicting results from general education research. A group of 127 sixth grade students completed a mathematical critical thinking test and a mathematical problem-posing task. The results showed that students' total score in the mathematical critical thinking test was a statistically significant predictor of their problem-posing fluency and problem-posing flexibility scores. However, students' total score in mathematical critical thinking test did not predict their problem-posing originality score. The results of this study support the position that originality is a more unique characteristic of mathematical creativity.

*Key words: Critical thinking, Problem posing ability*

## INTRODUCTION

The importance of critical thinking and problem-posing in mathematics is widely acknowledged. Several studies have been conducted to define and assess problem-posing ability (e.g., Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005), as well as to investigate the ways that problem-posing contributes to students' mathematical learning. In addition to this, several researchers investigated the relationship between problem-posing ability and mathematical creativity (e.g., Silver, 1997), as well as the relationship between problem-posing ability and mathematical knowledge (e.g. Van Harpen & Presmeg, 2013). Although much work has been done in this area, little attention has been given to the relationship between problem-posing ability and critical thinking in mathematics. Although there are claims that problem-posing activities can be used to identify and enhance critical thinking in mathematics (e.g., Bonotto, 2013), there is no clear evidence about the nature of this relationship. This is what is addressed in this paper.

## THEORETICAL BACKGROUND

### Definition of mathematical problem-posing ability

Mathematical problem-posing ability is defined as the ability to generate new problems and re-formulate given problems (Silver, 1994). Based on this definition, mathematical problem-posing is a creative activity and can be used to evaluate mathematical creativity (Silver, 1997). Since problem posing is a creative activity, the characteristics of creativity, fluency, flexibility and originality, can be used to describe students' ability to pose problems (e.g., Silver, 1997). Problem-posing fluency is the number of problems posed by a person. Problem-posing flexibility is the number of different categories of problems posed by a person, while problem-posing originality is a relative score since based on the rarest of problems posed by a person (e.g., Silver, 1997).

Moreover, Stoyanova and Ellerton (1996) defined problem-posing as the procedure to synthesize mathematical problems based on an individual's mathematical experience and interpretation. Using this definition Bonotto (2013) argued that problem-posing gives the opportunity to a person to interpret and critical analyse data.

At the same time, some researchers argued that mathematical problem posing is part of problem solving and the quality of problems that students posed is a predictor of students' problem solving performance (Kilpatrick, 1987). Moreover, problem posing activities can be: free (when students asked to pose problems without restrictions), semistructured (when students asked to pose problems based on given images and diagrams) and structured (when students asked to pose problems by translating given problem) (Stoyanova & Ellerton, 1996).

## Definition of mathematical critical thinking

Jablonka (2014) argued that there is no clear definition of critical thinking in mathematics. In the literature, general mathematical problem solving skills were usually connected with critical thinking (Jablonka, 2014). Moreover, "The Critical Thinking Consortium" (2013) defined critical thinking in mathematics as the ability to make reasonable decisions or judgments. In the present study, critical thinking in mathematics is defined as the ability to analyze, connect and evaluate data according to criteria or logic in order to take a decision or solve a problem. This description is based on the definition of critical thinking given by Iowa Department of Education (1989).

## Relationship between mathematical problem-posing and mathematical critical thinking

In general education research, there are conflicting results regarding of the relationship between critical thinking and creative thinking. For instance, Baker, Rudd and Pomeroy (2001) found that the two constructs are not closely connected, while the Iowa Department of Education (1989) argued that there is a lack of absolute boundaries between critical and creative thinking. Moreover, the Five Colleges of Ohio Creative and Critical Thinking Project (2009) based on a comprehensive literature review claimed that there are common behaviors between critical thinking and creative thinking (elaboration, complexity, synthesis, abstraction, simplification), but there are unique behaviors for each type of thinking. For example, imagination, invention, divergent thinking are unique characteristic of creative thinking, while analysis, interpretation, classification, logic are unique characteristics of critical thinking.

Regarding the relationship between mathematical problem-posing ability and critical thinking, which is the focus of this paper, there are claims that problem-posing tasks can be used to identify and improve critical thinking in mathematics (e.g., Bonotto, 2013), but this relationship is not clear based on the limited research findings. For example, Bonotto (2013) found that when fifth grade students were asked to pose problems based on a zoo park advertisement, they discussed the validity of the problems, the different restrictions and decided whether the problem could be solved. In other words, they used critical thinking skills to pose problems.

# THE PRESENT STUDY

## The purpose of the study

The purpose of this study is to investigate whether critical thinking in mathematics is related to mathematical problem-posing ability. More specifically, the present study addresses the following questions: (a) To what extent does students' mathematical critical thinking relate to students' performance in mathematical problem-posing? and (b) Is mathematical critical thinking a predictor of problem-posing fluency, flexibility and originality?

## Participants and instruments

One hundred twenty seven sixth grade students participated in this study. These students were studying in seven primary schools in Cyprus, both in rural and urban areas. All students completed a mathematical critical thinking test and a mathematical creativity test. The time allocated to complete the two tests was 70 minutes. The idea for the development of these tests is described in Sophocleous and Pitta-Pantazi (2015). The study presented here is part of a large scale study where 802 students took part. These are the preliminary results of the study with a sample of 127 students.

The mathematical critical thinking test included five tasks: three multiple-choice tasks and two short answer tasks. These tasks required from students to apply critical thinking processes such as the processes of analyzing, connecting and evaluating (Iowa Department of Education, 1989). For example, students were asked to give the common characteristic of a group of given shapes by analyzing and comparing the given shapes or to find the connection of a given bar graph with a statement or to evaluate data in order to reach a decision.

The mathematical creativity test included four tasks: one problem-posing task and three multiple solution tasks. These tasks were selected because according to a number of researchers (e.g., Leikin, 2009; Silver, 1997) they provide a suitable instrument for the measurement of creativity. These tasks required for their solution the application of the creative processes of imagination, synthesis and elaboration (Iowa Department of Education, 1989). For the purpose of this study, students' responses in the semistructured problem-posing task were used. The problem-posing task required from students to write as many different questions as possible which could be answered by the information provided by the given double bar graph. Students were also asked to think of solutions that nobody else would think of (see Figure 2 in Sophocleous & Pitta-Pantazi, 2015). To measure students' problem posing ability, three dimensions were evaluated: fluency (number of correct/feasible problems posed by a student), flexibility (number of different categories of questions posed by a student) and originality (calculated based on ratio of appearance of each type of question to all the type of questions). The different categories of questions were coded according to the three levels of graph comprehension proposed by Curcio (2010): reading data, reading between data and reading beyond data and categories of additive and multiplicative problems: combine and compare (Fuson, 1992).

Based on the data, the questions that required reading beyond the data were posed only by 6 students out of 127. While the questions that required reading of the data and reading between the data were posed by a large number of participants. Specifically, they posed 53 questions that required reading of the data and 366 questions that required reading between the data. These questions appear in Cyprus mathematics textbooks. Moreover, the number of posed questions with combination structure is bigger than the number of posed questions with comparison structure (184 and 120 respectively). In addition to this, the number of posed problems which associate two structures: comparison and combination or analogy and combination are fewer than the problems with one structure (67 and 304 respectively). In addition to this, questions with more complicated structure (negative structure or multiplicative structure) posed by few students (2 and 1 respectively). Therefore, the rarest questions posed by participants were those that required reading beyond the data and their structure was more complicated.

**RESULTS**

Descriptive analysis was used to present students' performance in mathematical critical thinking test and in problem-posing activity in terms of fluency, flexibility and originality. Table 1 shows the results of this analysis. Specifically, Table 1 shows the means, standard deviations, minimum and maximum scores of participants' performance in the critical thinking test and in problem-posing activity, in terms of fluency, flexibility and originality.

According to the data presented in Table 1, students' performance in critical thinking test was slightly over 0.60. Participants were able to solve more than half of the tasks of the test. Students' problem-posing fluency average score was 7.54 and the maximum score was 27. In other words, the average number of questions that participants posed was approximately 8, while there were students who posed up to 27 questions. Participants' problem-posing flexibility average score was 3.50 and the maximum score was 9. This means that the average number of different questions that participants posed was approximately 4. Participants' problem-posing originality average score was 1.81, with maximum score 4. The scores in the problem-posing terms showed that students were able to produce a number of different questions based on the given double graph. Moreover, there were statistically significant differences in students' performance in the mathematical critical thinking test and in their problem solving fluency and flexibility. Their performance in the critical thinking test is higher than students' scores in problem-posing fluency and flexibility. However, there were no statistically significant differences between their scores in the critical thinking test and in the problem posing originality. This result is explained in more depth below.

| | Mean ($\overline{X}$) | SD | MINIMUM | MAXIMUM |
|---|---|---|---|---|
| Critical Thinking | 0.62 | 0.16 | 0.22 | 0.97 |
| Problem-Posing Fluency | 7.54 | 4.58 | 0 | 27 |
| Problem-Posing Flexibility | 3.50 | 2.21 | 0 | 9 |
| Problem-Posing Originality | 1.81 | 1.02 | 0 | 4 |

Table 1: The descriptive data of participants' performance in mathematical critical thinking test and in problem-posing activity in terms of fluency, flexibility and originality

Pearson correlation (r) was used to explore the association between critical thinking test score and problem-posing fluency, flexibility and originality scores. According to the data, critical thinking test score was significantly related to problem-posing fluency and flexibility score ($r_p$=0.31 and $r_p$=0.24 respectively). While participants' critical thinking test score was not related to their problem-posing originality.

To investigate in more depth the association between critical thinking and problem-posing fluency and flexibility, we used regression analysis. This analysis was conducted with independent variable the participants' total score in mathematical critical thinking test and with dependent variable their scores in each of the dimensions: fluency, flexibility and originality in problem-posing. The results are presented in Table 2. Using the enter method a significant model emerged for problem-posing fluency ($F_{1,\ 75}$=7.67, $p$=0.007) and for problem-posing flexibility ($F_{1,\ 75}$=4.76, $p$=0.03), while the analysis does not give a significant model for problem-posing originality ($F_{1,\ 75}$=1.08, $p$=0.30).

From Table 2, it appears that mathematical critical thinking score can be a statistically significant predictor of participants' problem-posing fluency ($\beta$=0.31, p-0.007) and problem-posing flexibility ($\beta$=0.24, p=0.032). It can explain 10% and 6% of the variance of participants' problem-posing fluency and flexibility respectively. The positive sign of beta coefficients showed a positive effect of critical thinking score on participants' problem-posing fluency and flexibility. In other words, students with high score in mathematical critical thinking test have higher fluency and flexibility than students with low score in mathematical critical thinking test.

| | Problem-posing fluency | | Problem-posing flexibility | | Problem-posing originality | |
|---|---|---|---|---|---|---|
| | $B$(SE) | $\beta$ | $B$(SE) | $\beta$ | $B$(SE) | $\beta$ |
| Total score in mathematical critical thinking test | 0.38 (0.14) | 0.31** | 0.14 (0.06) | 0.24* | 0.03 (0.03) | 0.2 |

$R^2$=0.10 for problem-posing fluency, $R^2$=0.06 for problem-posing flexibility *p<0.05, **p<0.01

Table 2: Regression analysis with dependent variables problem-posing scores in the dimensions: fluency, flexibility and originality and independent variable total score in mathematical critical thinking test

## DISCUSSIONS

The purpose of this study was to investigate the relationship between mathematical critical thinking and mathematical problem posing ability. These preliminary results showed that mathematical critical thinking correlates and predicts students' problem-posing fluency and flexibility, but not their problem-posing originality.

First, the predictive role of mathematical critical thinking to problem-posing fluency and flexibility can be explained. Students need to use critical thinking processes (analysis, connection, evaluation) to produce a number of problems (fluency) and different problems (flexibility). Specifically, they analyse the given information, find main ideas to write problems and recognize patterns to produce many problems. In addition to this, they evaluate the problems that they already wrote seek similarities or differences among them and try to produce different problems. This result is in accord with Bonotto's (2013) study in which students used critical thinking skills to produce new problems. Bonotto (2013) did not specify if students' critical thinking skills influence all the dimensions of problem-posing ability. The result of this study extent Bonotto's (2013) study result.

However, the results of the study showed that mathematical critical thinking score did not predict problem-posing originality. In other words, it seems that participants did not use critical thinking skills to produce the original questions; these questions which require reading beyond the data and more complicate synthesize data. Probably these participants used creative thinking skills such as imagining and synthesize, to produce the rarest questions. This result is in accord to the position that originality is rather an internal characteristic of creativity (Leikin, 2009). Originality seems to be a unique characteristic of mathematical creativity. Further research is needed to verify this claim in relation with other mathematical creativity tasks and in other educational systems since the results of this study cannot be generalized.

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# SELF-SELECTED LEARNING COMMUNITIES FOR PROBLEM SOLVING IN ONLINE MATH COURSES

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**Abstract.** *The purpose of this study is to examine the impact of students' self-selection of their learning community on the depth of learning of cognitively challenging problem solving tasks. Students in multiple sections of an on-line math course were engaged in mathematical problem solving within an instructor-selected (IS) group. Cognitive tasks were selected in order to challenging students so they would seek out the IS group. The intent was for students to develop a relationship with others in IS group in order to engage in dialogic inquiry about how to solve the problems. Using grounded theory and situated learning theories, the researcher found that students who selected their own learning communities, from outside the course, showed a depth of learning that was comparable to that of students who engaged solely in IS groups.*

*Key words: problem solving, learning community, dialogue, cognitively challenging tasks, online, groups*

## INTRODUCTION

Studies have found that students benefit from dialogic inquiry when they encounter cognitively challenging mathematical problem solving. When learners talk about their understanding in the process of problem solving, it helps them make sense of what they're learning. Whether the student's mathematical foundation is strong or tenuous, sharing partially developed ideas is a way to further the student's thinking and understanding. Dialogue provides opportunities to test and consider ideas. The exploration of the process of partnering with someone else to generate opportunities for dialogue and inquiry is explored in this study.

## LITERATURE REVIEW

The body of research on the importance of discourse among students in the learning of mathematics in the classroom is rich and varied. When learners talk about their thinking during problem solving, it helps them make sense of what they're learning.

Dejearnette & Gonzales (2016) linked their own study of dialogic inquiry in the high school geometry class back to earlier studies that point to the need of dialogic inquiry when learning mathematics. Their work supported studies which found that dialogic inquiry as part of student group work is critical in learning to do mathematics.

In a summary of the importance of talk in learning, Burns (2013) identified five reasons for engaging in talk about learning math. She indicated that talk reveals understanding or misunderstanding, boosts memory, increases reasoning, supports language development, and helps develop social skills.

In order to encourage dialogic inquiry, learners must be engaged in cognitively challenging tasks. When the tasks that students are engaged in are sufficiently challenging and require critical thinking, significant levels of learning are achievable. Cai and Lester (2001) found that the use of such tasks improves the mathematical learning of students. These tasks or problems require students to reason and communicate mathematically to further their mathematical understanding.

Completing cognitively challenging tasks requires the learner to struggle. Engagement in productive struggle is an opportunity for learners to further their understanding (Bjork & Bjork, 2011). When they're engaged in problem solving tasks, students have opportunities to struggle productively and to learn more. (Hiebert & Wearne, 2003).

Leon and Castro (2016) explored the use of cooperative learning groups as a pedagogical tool for engaging students in ways that would shape their communication and interactions to create a more positive environment. The focus of their work was to create an environment that was more like a respectful community.

Chen and Chang (2011) examined the social learning networks of students and tapped into them in order to form more productive cooperative learning groups improved levels of engagement in problem solving. The process of exploring patterns of social interaction in order to form groups proved beneficial to the learners. It improved the level and type of engagement which in turn, improved the learning outcomes.

## THEORETICAL FRAMEWORK

The background and framework for this study is based in learning mathematics through exploration and application. It is grounded in sociocultural activity and situated learning theories (Creswell, 2013, Vygotsky, 1978). The researcher used grounded theory to examine the data that emerged over a ten-year period regarding the development of learning communities by students who were also connected to cooperative learning groups in the on-line setting. There were a few emergent themes that were explored and analyzed.

## DESIGN

This study examines the range of ways classroom learners engaged in dialogic inquiry about mathematical problem solving both in and outside the formal setting of an on-line class, over 10 terms. The researcher used grounded theory to analyze themes that emerged regarding decisions made by learners about how to engage socially in exploration of the mathematical problems given in a course. In each class, there was a planned approach by the instructor to bring groups of two to four people together in order to engage in dialogic inquiry about mathematical problem solving.

The problems that the students in the math course were asked to solve come from algebra, geometry, number theory, statistics, and probability. Many would be familiar to those who teach mathematical problem solving. An example of one of the algebra problems is the Emperor's Banquet.

> You have been invited to the emperor's banquet. The emperor is a rather strange host. Instead of sitting with his guests at a large round dining table, he walks around the table pouring oatmeal on the head of every other person. He continues this process, pouring oats on the head of everyone who has not had oats until there is only one person left. (Remember he is skipping every other person who has not been hit as he goes around.) You will know when you arrive where the first seat is to get the oats and that the emperor always rotates to the left with his oats. You do not know in advance how many people are there. Devise a system that will tell you in

what seat to sit in order not to get "oatmealed" once you know how many people are at the banquet.

Students were asked to begin solving the problem individually. Then they were to come together with their group to discuss and further explore how to solve the problem. Students were not allowed to simply tell others how to solve the problem if they knew. After taking about ten days to work on and discuss the problems, students submitted their written solution and those attempts they tried that didn't work. They were to indicate the level of challenge of the problem. And they were to provide real-life connections to each of the problems solved. Table 1 describes the type of groups formed, along with key characteristics or attributes of each.

| *Type of Group* | *Composition of Group* | *Level/type of mathematical task* | *Dialog Structure* | *Assessment Type* |
|---|---|---|---|---|
| Instructor-selected group | 2-3 graduate classmates | Problem-solving tasks which challenged math teacher | On-line dialogue | Individual Written Explanation with solution |
| No group | None; individual | Problem-solving tasks which challenged math teacher | None | Individual Written Explanation with solution |
| Self-selected team | Either colleagues or family | Problem-solving tasks which challenged math teacher | Face-to-face | Individual Written Explanation with solution |
| Facebook | Facebook friends who chose to explore problems | Problem-solving tasks which challenged math teacher | On-line dialogue | Individual Written Explanation with solution |
| Math expert partner | Student selected college math professor | Problem-solving tasks which challenged math teacher | Face-to-face | Individual Written Explanation with solution |

Table 1: Type of Group Formed, with other characteristics or attributes

The different inquiry groups that were developed range from no group, to instructor-selected group, to self-selected. The relationships among the type of group that was formed, the mathematical tasks in which the group was engaged and the venue for dialogue are discussed. The results of students' attempts at understanding the mathematics are included. Conclusions about the perceived effectiveness of working in the group are shared, along with limitations.

## DISCUSSION

The discussion will focus on the type of group, their dialogue and the resulting learning, by group. The instructor-selected groups were chosen based on how those same students engaged during the course that was offered the prior term, which was taught by the same instructor. More engaged learners would be distributed across the groups. Students who were quiet or less engaged in the prior term would be distributed across groups. This distribution was intended to avoid the creation of groups that didn't communicate. The instructor embedded discussion boards for each group that were dedicated to discussing the problem set that was assigned, usually including five or six problems, of which each student could eliminate any one. The assessment that was given was individualized. Each student was to write up an individualized solution set, including what they tried that didn't work, whether the problems challenged them, and real-life connections.

In the instructor-formed groups, students acknowledged that they achieved greater success at mathematical problem solving when they engaged in dialogue about how to solve the problem, than when they didn't devote the time to dialogue and worked individually. This was the case whether the students were initially on a right path toward solving the problem or not. There were instances during the term when a learner was more engaged in the dialogue than at other times. That difference was noticeable in the written assessments of understanding. In some instances, the student acknowledged the limited engagement when submitting the assessment. When the student was able to re-engage, the level of work improved.  Those students who didn't engage in dialogue but then tried to write up the assessment often had big gaps in understanding. In one case, the learner regularly commented in supportive ways regarding others' ideas without adding substantively to the dialogue. That person's reflections had gaps in understanding and were inaccurate because there was not enough substantive engagement to support the development of understanding.

In the instance where students chose to work in isolation to solve the problems, the level of success was low. The students were not penalized for not working in a group, so the grade was only reflective of their level of learning.  The individual work was very limited. The pattern was that students would try the first seemingly logical solution and write it up for submission. The assessments reflected little understanding beyond the initial idea. There was no evidence of thinking of other ways to solve a problem or of limitations regarding the approach that had been tried. Critical thinking about whether the solution to the problem was reasonable was rarely in evidence. The individual explanation was limited and often the solution was wrong.

When students created local learning groups of their own, either comprised of family members, or a partner, there was an interest on the part of the student in sharing what was being done in the class. There was also an interest on the part of others in knowing what the student was doing in the course. When the problems were submitted for evaluation in the course, the students exhibited a level of understanding of the problem that was not different from those who worked in instructor-selected teams. Within the self-selected teams, students were successful at solving the problem and analyzing misinterpretations. An additional benefit was a level of enthusiasm about the fact that what was being learned was being shared beyond the course, with people who were important to the individual. This information came out during discussions, in the problem set write-up, or in an evaluation of how the instructor-selected cooperative learning groups were working. In the instances where the student selected her or his own team, the student would articulate the enthusiasm regarding the fact that people around her or him were interested in exploring what the student was learning. Some of the reflections captured curiosity or excitement about delving into a new set of problems each week. The learning goals were met when the student engaged in doing mathematical problem solving with his or her own self-selected team.

The student who posted the problems on Facebook apologized for doing so when the information was shared at the end of the term. It was as if there was something was wrong with engaging with the Facebook group about the work to be done for the course. It wasn't clear what percentage of the student's Facebook group engaged in talking about or doing the problems with her. She indicated that the group was comprised of family, friends, colleagues, and some of her students. It was clear that the learning goals were met by the student when she worked with her Facebook group. Her assessments reflected a depth of understanding that was comparable to the work of her classmates.

The student who chose a math expert as her partner did this during the first term that the course was offered. She was the only person in that term of the course who worked with a partner. The other students worked individually, although they were told to choose two classmates and form a group. Although the student didn't choose to work with two classmates, she did choose a partner. She was the only student who consistently achieved the learning goals for problem solving during that term. Her partner was her father, who was a college-level math instructor. She had success with the problems. Her reflections showed a depth of understanding regarding how to analyze and solve the problems. She was able to think critically about the reasonableness of solutions. She did indicate that she and her father were both sufficiently challenged by the problems to make the process of solving them difficult.

## CONCLUSIONS

When students in the on-line math course formed their own inquiry groups with family members or other interested individuals, there was not the structured interaction that had been built into the on-line setting. In several of the instances, the learner developed an approach to the collaboration that was quite different than the original design. In some instances the learner decided not to interact with others. The student tried to solve problems independently. The ways that learners engaged in dialogue included working in instructor-created online groups; partnering with a math expert who was not part of the course; posting problem situations on Facebook, then exploring solutions with Facebook friends; working with teacher colleagues; and choosing a child as a math exploration partner for engaging in mathematical tasks. The one approach that yielded the worst results for the learner was working in isolation. When the student worked with others who were not in the course, they found more success than those who worked in isolation. The other groups, whether instructor-selected or student-selected yielded results that were not different, based on the type of group.

There were instances over the course of the term where students weren't able to collaborate with their groups. In several instances, there was an awareness that the resulting work was not as strong. Martha indicated that she worked individually during the Algebra problem solving module which fell during the last weeks of her teaching term. She wrote that she was not able to fully make sense of the Emperor's Banquet problem, due to her work in isolation. She indicated that she thought she'd have been successful with the problem, had she been in dialogue with her group.

There were also instances where students' successful learning experiences with dialogue were causing them to generate questions about how to use dialogue in their own classrooms. For example, Cara said about the Geometer's Sketchpad, "I also would like to know how the teachers would create a discussion while using this program. It seems to me that the students would work individually. I think students working individually is fantastic however, it is also important to have an engaging discussion with the students. I am wondering if the student would have a full class discussion about their buildings or work in small group discussion."

One student quoted one of our texts in response to one of the assignments. "If they can't explain, it means that they're still in the process of trying to understand. Trying to explain can help them understand what they know and don't know." (Sakshaug, 2014, p. 47)

When student worked in the IS groups, the learning goals were achieved. When a student formed her or his own group, there was evidence of engagement and the development of understanding, based on the assessment score. It is not known whether the student would have learned more had the group been comprised of classmates, rather than family members or members of the Facebook community. It is likely that there is a higher level of mathematical understanding on the part of a group of mathematics teachers than there is among a group of people on a Facebook page or someone's children. However, in the instances where the math teacher worked with a group that was not comprised of other math teachers, there was successful learning of the content. In each instance, the student's assessments reflected an understanding of the content that was comparable to their classmates' learning. In the instances where students chose to work in isolation or weren't able to engage with their group in a timely way, the assessments reflected a very limited understanding of the mathematical problem solving, as reported in Sakshaug, 2010. When students selected their own learning communities, the results of the engagement and discussion with the group they selected did not have a noticeably different result in terms of their learning of the mathematical problem solving.

Although there is more study to be done about how groups are comprise, the result of this study indicate that in the instances where groups were formed by the student, there was no interference with learning. When students selected their own group, the student was able to successfully complete the mathematical problem solving.

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# THE ROLE OF THE TEACHER-INVENTOR IN THE DIFFERENTIATED LEARNING ENVIRONMENT

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***Abstract.*** *The teacher-consultant as the usual role of the teacher in the differentiated learning environment is not attractive or offering any development opportunities for teacher. The more appropriate role for the teacher conducting differentiated instruction, especially of gifted students, is that of the teacher-inventor, who can think up tasks on the spot, is always a few steps ahead, experiments with his students and accepts challenges from them.*

*Key words: differentiated instruction strategies, gifted students, teacher-inventor, olympiad problems in math coursebooks, math contest*

## DIFFERENTIATED INSTRUCTION STRATEGIES

Differentiated instruction in math is one of the most effective approaches to teaching gifted students in the regular classroom. Differentiation suggests that the teacher symbolically divides the students into groups based on their skills and knowledge level (i.e. level groups) and provides each group with tasks at varied levels of difficulty. It is believed that through the use of such tasks the teacher promotes the students' learning development in accordance with their abilities. All math coursebooks designed for Estonian students provide them with material of an average and higher level of difficulty.

Despite the evident advantages of differentiated instruction, it likewise has certain disadvantages. Its main drawbacks include students' separateness from their classmates, the lack of a common goal among the students studying together, a varying level of motivation among the level groups, and a less holistic approach to the teaching which, according to the current school curriculum, consists in guiding students towards exploratory activity.

The author, who for the last 10 years has been teaching math at school, working with gifted students at the republican level and compiling the national olympiad and exam papers in Estonia, is of the opinion that the role of the teacher-consultant in the differentiated learning environment is not attractive or offering any development opportunities for him as a teacher.

The math teacher must constantly and actively be involved in a thinking process, be inventive and creative, that is, be the one he wants his students to be. Therefore, it is absolutely necessary for the teacher to stay within the learning process for the sake of his self-development instead of managing it externally by means of giving his students pre-selected learning tasks. According to the author, the more appropriate role for the teacher conducting differentiated instruction, especially of gifted students, is that of the teacher-inventor, who can think up tasks on the spot, is always a few steps ahead, experiments with his students and accepts challenges from them.

The teacher-inventor can apply differentiated instruction to the coursebook tasks exclusively, which will, in turn, serve as a starting point for inventing a great deal of math problems of a varied level of difficulty. The author has named this method of presenting learning material "some out of one". Almost every coursebook task can be both simplified (finding “closer” elements, direct counting, using specific cases, checking solutions, replacing data, solving by analogy etc.) and complicated (finding “further” elements, solving inverse problem, different ways of solving, considering the general case, open questions etc.). As a result, the whole class, both its weaker and stronger students, solve the initial math problem and its different variations. Knowing the point of the initial problem, every student takes active part in analyzing the solutions suggested by their classmates by listening to and understanding their explanations, asking questions, and talking to the teacher and the classmates. At the end, the teacher always gives an overview of the work done and the results obtained, supplements the students’ solutions if needed, gives solutions of the problems the students have not managed to solve, answers the students’ questions and solves problems suggested by his students.

With a view to giving examples of using the above-stated method in the classroom environment, the author has chosen the first course of extensive mathematics called "Expressions and Numerical Quantities", which is taught in secondary schools of Estonia. According to the national curriculum at the end of this course, students are be able to explain the properties of the sets of natural numbers and integers, rational, irrational and real numbers, mark the regions of real numbers on a number line, perform operations with powers and roots, transform rational and irrational expressions and solve problems with applied content (including percentage problems).

As seen from its learning outcomes, the course teaches the basics of algebraic transformations. The majority of the coursebook tasks dealing with the properties of powers and roots and transformations of algebraic expressions require calculating numerical expressions, factoring and simplifying expressions. What is more, a large proportion of the coursebook tasks, especially those of the average level of difficulty, solely require the direct use of powers and roots, algebraic formulas or standard algorithms for simplifying expressions.

For example, the topic "Properties of square roots" presented in the coursebook is introduced through a variety of tasks ranging from square root calculation $\sqrt{0{,}04}$, $\sqrt{(-4)^2}$, $\sqrt{0{,}04 \cdot 10^8}$ and the tasks alike to simplification $\sqrt{\left(\sqrt{2}-3\right)^2}$, $\sqrt{32}-3\sqrt{2}$, $\left(3\sqrt{2}+\sqrt{3}\right)^2$, $\left(3-2\sqrt{2}\right)\left(3+2\sqrt{2}\right)$ and the other similar tasks. The coursebook unit built around the topic of the properties of $n^{\text{th}}$ roots is designed almost identically to that focusing on identifying the properties of square roots. It goes without saying that many students experience difficulties when solving these kinds of problems on their own, especially, if they have got gaps in the subject knowledge gained in basic school, whereas more talented students do not benefit from solving them every successive lesson. For them, the tasks of an average level of difficulty provided in the coursebook fall into the category known as "glanced and found the solution" – at best, they agree to solve some of them for the practice purpose. Among the problems of a higher level of difficulty there is a limited amount of more informative problems that can generate interest in more talented students. These include square root simplification tasks $\sqrt{7-4\sqrt{3}}$, $\sqrt{4+2\sqrt{3}}-\sqrt{4-2\sqrt{3}}$ and $\sqrt{17-4\sqrt{9+4\sqrt{5}}}$ and prove-the-equality tasks like $\sqrt[3]{5\sqrt{2}+7}-\sqrt[3]{5\sqrt{2}-7}=2$. However, the main downside of these problems is that they are all solved by one solution (based on the example given in the coursebook), which means that they are algorithmic, too.

Thus, the teacher who sticks to the coursebook material is forced to give their students a so-called algorithm package instead of developing the skills of algebraic transformations in more talented students by enabling them to search for any possible solutions to the given problems.

One of the ways of enriching the coursebook material on the given topic is to develop a problem taken from the coursebook to the theory of this problem, that is, to get "some out of one". For example, having chosen a standard math problem consisting in simplifying an expression containing square roots $\sqrt{6-\sqrt{11}} \cdot \sqrt{6+\sqrt{11}}$, the teacher-inventor can develop it in two directions (both in the direction of making it simpler and in the direction of making it more complex).

For less talented students, the teacher-inventor first creates the route for solving the problem chosen and then immediately suggests some possibilities for using the results obtained.

To give an example, for less talented students, the process of studying the properties of square roots can consist of the following components:

1. understanding that the formula for the difference of two squares works in both directions: $(a-b)(a+b) = \ldots$, $a^2 - b^2 = \ldots$;
2. understanding that this formula is applicable both to literal and numeric values: $\left(a-\sqrt{b}\right)\left(a+\sqrt{b}\right) = \ldots$, $\left(6-\sqrt{11}\right)\left(6+\sqrt{11}\right) = \ldots$;
3. studying the conditions for finding square roots: $\sqrt{6-\sqrt{11}}$, $\sqrt{\sqrt{11}-6}$, $\sqrt{\left(\sqrt{11}-6\right)^2}$;
4. simplifying expressions with the use of root properties: $\sqrt{6-\sqrt{11}} \cdot \sqrt{6+\sqrt{11}}$, $\sqrt{\left(6-\sqrt{11}\right)^2} - \sqrt{\left(\sqrt{11}-6\right)^2}$, $\frac{\sqrt{6+\sqrt{11}}}{\sqrt{6-\sqrt{11}}} + \frac{\sqrt{6-\sqrt{11}}}{\sqrt{6+\sqrt{11}}}$;
5. searching for different solutions to one problem: $\left(6+\sqrt{11}\right)^2 - \left(6-\sqrt{11}\right)^2$, $\sqrt{\left(6-\sqrt{11}\right)^4} \cdot \sqrt{\left(6+\sqrt{11}\right)^4}$;
6. finding different ways of eliminating irrationality in denominator: $\frac{\sqrt{11}}{6-\sqrt{11}} + \frac{6}{\sqrt{11}+6}$, $\frac{\sqrt{6+\sqrt{11}}}{\sqrt{6-\sqrt{11}}}$;
7. checking by calculator the solutions obtained.

More talented students are definitely offered those problems whose solutions do not lie on the surface and those questions which are more non-standard and require some analysis on their part. The main emphasis must be laid upon the search for the right way of solving a problem and for the more rational and beautiful solution. The comparison of different solutions and assessment of their beauty are possible not only for olympiad-level problems but also for 'school-type' problems. Thus, the set of math problems for more talented students can look as follows:

1. comparing expressions containing roots: which is larger, $\sqrt{6+\sqrt{11}}$ or $6-\sqrt{11}$, $\sqrt[4]{6+\sqrt{11}}+\sqrt[4]{6-\sqrt{11}}$ or 3;

   $$\left(\begin{matrix} \sqrt{6+\sqrt{11}} & ? & 6-\sqrt{11} & & 13\sqrt{11} & ? & 41 \\ 6+\sqrt{11} & ? & 47-12\sqrt{11} & & 1859 & > & 1681 \end{matrix}\right)$$

2. finding the exact value of the expression $A=\sqrt{6-\sqrt{11}}+\sqrt{6+\sqrt{11}}$;

   (searching for the idea of the solution: $A^2=22$, where $A=\sqrt{22}$)

3. finding all solutions: find all integers $a$, so that the value of the expression $A=\sqrt{6-\sqrt{a}}+\sqrt{6+\sqrt{a}}$ is any integer;

   ($A^2=12+2\sqrt{36-a}$ is an integer, so $a=32$ is the only possible value)

4. finding different ways of solving equation applying the principle of comparison: is it possible to find the value of the integer $a$ which is different from number 6, so that $\sqrt{a-\sqrt{32}}+\sqrt{a+\sqrt{32}}=4$ is true?;

5. searching for different ways of eliminating a double square root: $\sqrt{6+\sqrt{32}}$, $\sqrt{6+\sqrt{11}}$;

   $$\left(\begin{matrix} \sqrt{6+\sqrt{32}}=\sqrt{2+4+2\sqrt{2\cdot 4}}=\sqrt{\left(2+\sqrt{2}\right)^2}=2+\sqrt{2} \\ 6+\sqrt{32}=6+4\sqrt{2}=\left(\mathrm{a}+\sqrt{2}\mathrm{b}\right)^2=\mathrm{a}^2+2\sqrt{2}\mathrm{ab}+2\mathrm{b}^2 \quad\Rightarrow\quad \begin{cases}\mathrm{a}^2+2\mathrm{b}^2=6\\ \mathrm{ab}=2\end{cases} \\ \begin{cases}\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}=\sqrt{22}\\ \sqrt{6+\sqrt{11}}-\sqrt{6-\sqrt{11}}=\sqrt{2}\end{cases} \quad\Rightarrow\quad \sqrt{6+\sqrt{11}}=\frac{\sqrt{22}+\sqrt{2}}{2} \end{matrix}\right)$$

6. getting the solution in its general case: work out a formula for eliminating the double square root in the expression $\sqrt{a+\sqrt{b}}$, where $a\geq\sqrt{b}$;

7. solving proof problems: prove that $\sqrt[4]{6+\sqrt{11}}\cdot\sqrt[3]{6-\sqrt{11}}>\sqrt{5}$.

   $\left(\sqrt[4]{6+\sqrt{11}}\cdot\sqrt[3]{6-\sqrt{11}}=\sqrt{5}\cdot\sqrt[12]{6-\sqrt{11}}>\sqrt{5}\right)$

The problems presented above are not at all special. The teacher having a sufficient experience in this subject matter can invent a big variety of them. These problems meet the principle of differentiated instruction, which states that every student has the opportunity to choose their starting point and move forward with every new task. Besides the external resemblance to the original problem, the majority of the problems set forth by the author make use of the results obtained when solving the original problem. Hence, they are - even though artificially - united with a common goal and are more attractive for students. These problems underlie the emergence of the new method of presenting learning material which is particularly beneficial for more talented students – as it enables them to generate a great deal of ideas about new problems and set forth new hypotheses to solve and prove them either on their own or with the teacher. Even more complex problems do not seem infeasible to them. As to less talented students, having received basic knowledge when dealing with easier tasks, they get a chance to grasp how more complex solutions can be developed. Thus, less talented students are also engaged in the process of studying the one math problem.

**ENRICHING THE COURSEBOOK MATERIAL WITH OLYMPIAD MATH PROBLEMS**

The original problem has been chosen purposefully, after having considered the results of the Estonian Republican Mathematical Olympiad of the year 2017. In recent years, the sets of six math problems designed for secondary-school students have had two 'school-type' problems. Their aim is to improve emotional wellness of those students who for some reason or other cannot succeed in solving olympiad problems. The sets of the year 2017 contained three 'school-type' problems: $10^{th}$ graders were supposed to find all pairs of integers $(a, b)$, for which the equality $324^{a+b} = 3^{a-3} \cdot 2^{a-b} \cdot 4^{b-4}$ is true, $11^{th}$ graders were supposed to compare the numbers $7 + \sqrt{37}$ and $3\sqrt{19}$ without using a calculator, and $12^{th}$ graders were supposed to find all pairs of nonnegative integers $(a, b)$, for which the equality $\sqrt{a} + \sqrt{b} = \sqrt{217}$ is true. All the skills needed to solve these three problems could have been obtained when solving some of the problems that have been offered above. What is more, the three problems just described do not exceed in difficulty the ones considered much earlier in this text. However, despite this fact, only 19% of the $10^{th}$ graders, 33% of the $11^{th}$ graders and 15% of the $12^{th}$ graders participated in the Olympiad scored more than half of the points for solving these 'school-type' problems. First and foremost, these data indicate that the majority of the more talented Olympiad participants are not familiar with the easiest ways of solving such 'school-type' problems as comparison of roots containing expressions.

Coursebook-centred teaching and the teacher's adherence to solving calculative problems through algorithms only are the main factors leading to such unsatisfactory results. Students must develop in the course of solving every single problem. They must be offered the whole set of mathematical instruments they can master. This is feasible only by means of differentiated instruction and only under the supervision of the teacher-inventor.

There are a lot of ways of enriching the coursebook problems within the course titled "Expressions and Numerical Quantities" with math problems exhibiting features of olympiad ones – they are imperative given talented students in class. For example, studying the set of integers opens the doors to such an area of olympiad mathematics as number theory. Expression simplification is closely connected to the method of mathematical induction which is not included in the Estonian National Curriculum in Math.

Here is a short list of 15 standard olympiad problems, with each of them representing a big class of math problems, that can be considered in lessons aiming at developing students' understanding of expressions and numerical quantities by those students who have interest to them.

1. Prove that $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$ is an *irrational* number.
2. Find all integers $a$ such that $\frac{a^3 - 5a^2 + 20}{a - 1}$ is an *integer*.
3. Prove that 504 *divides* $a^9 - a^3$ for all integers $a$.
4. Prove that $237^{231} + 732^{132}$ is a *composite* number.
5. How many digits are in the result of $4^{102} \cdot 5^{201}$?

6. What number is smaller, $2^{2020}$ or $3^{303} \cdot 4^{404} \cdot 5^{505}$?
7. Prove that 90 divides $9^{32} + 3^{66}$.
8. Prove that $\sqrt{16 + 6\sqrt{7}} + \sqrt{32 - 10\sqrt{7}}$ is an integer.
9. What number is larger, $\sqrt{101} + \sqrt{104}$ or $\sqrt{102} + \sqrt{103}$?
10. Use basic *algebra formulas* to prove that $4^9 + 6^{10} + 3^{20}$ is a square number.
11. Use the *pairing up* method to prove that 32 divides $1^a + 2^a + 3^a + \cdots + 15^a$ for all odd integers $a \geq 3$.
12. Factor the expression $a^4 + a^2b^2 + b^4$ by *grouping*.
13. Use the *completing the square* method to find all pairs of positive integers $(a, b)$ such that $a^2 - 4ab + 5b^2 = 89$.
14. Prove by *induction* that 64 divides $3^{2a+2} - 8a - 9$ for all positive integers $a$.
15. Using a *telescoping series* find the sum $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ \ldots \ + \frac{1}{99\cdot 100}$.

According to the recently conducted survey among Estonian students and math teachers, the majority of the students questioned believed that the teacher and coursebook were the main sources of information, when learning math, while the majority of the math teachers surveyed reported using coursebook as the main teaching tool. At the same time, both the National Curriculum and the material of math coursebooks undergo continuous simplification. For this reason, with every new coursebook edition, differentiated instruction with the use of coursebook becomes more and more difficult to conduct. The author is of the opinion that although math coursebook is important, it must serve as an aid, not be the only teacher's tool. It is vital that the teacher not only solves math problems and teaches their students this skill, but also constantly impresses their student audience by thinking up something that might interest them. To the author's mind, the role of the teacher-inventor must gain its popularity in the modern school and this must be taken into account when training future math teachers at university level.

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# MATHEMATICS FROM OLD TIMES AND FOREIGN COUNTRIES – ENCOURAGING MATHEMATICAL CREATIVITY OF PRIMARY STUDENTS THROUGH ETHNOMATHEMATICS

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***Abstract.*** *Fostering mathematical creativity in primary school age is a very important but also a challenging task for teachers. Sternberg's Investment Theory of Creativity can give some general hints for this purpose. From the perspective of mathematics education, ethnomathematics could also have some specific potentials in this regard. In this paper, we present a workshop series "Mathematics from old times and foreign countries" combining both approaches.*

*Key words: mathematical creativity, ethnomathematics, Investment Theory of Creativity*

## CONCEPTUAL CONSIDERATIONS

### Investment Theory of Creativity

Creativity can be seen as a for a long time neglected research topic. By today, there is no commonly acknowledged creativity theory nor even a widely accepted definition of creativity. For instance, Sternberg und Lubart (1999) differentiate between mystical, pragmatic, psychodynamic, psychometric, cognitive and social-personality approaches to the study of creativity. In recent years, confluence approaches have additionally established based on the basic assumption that multiple components must converge for creativity to occur. An influential approach of this kind is Sternberg's Investment Theory of Creativity (e.g. Sternberg, 2006). It describes six distinct but interrelated resources:

*Intellectual abilities.* According to Sternberg, the confluence of three intellectual abilities is particularly important for creativity: the synthetic ability to see problems in new ways and to escape the bounds of conventional thinking, the analytic ability to recognize which of one's ideas are worth pursuing and which are not, and the practical–contextual ability to know how to persuade others of the value of one's ideas.

*Knowledge.* On the one hand, domain-related knowledge is regarded as an essential prerequisite for creative action (Silver, 1997; Weisberg, 1999). On the other hand, knowledge can hinder creativity by resulting in a static and closed perspective.

*Thinking styles.* They can be seen as preferred ways of using one's abilities but have to be distinguished from the abilities themselves. Especially important for creativity is a preference to think in novel self-chosen ways.

*Personality*. Certain personality attributes seem to be important for creative action. Amongst others, these attributes include self-efficacy, willingness to overcome obstacles, sensible risk tolerance, and willingness to tolerate ambiguity.

*Motivation.* Intrinsic, problem-focused motivation is also important for creativity. It is not an attribute of the personality but one decides in each case to be motivated depending on the current situation or the problem to be addressed.

*Environment.* For creative action one needs an environment that is supportive and rewarding of creative ideas.

Concerning the *confluence* of these six components, Sternberg states first that there may be thresholds for some components below which creativity is not possible regardless of the levels on other components. Second, compensation may partly occur in which a strength on one component compensates a weakness on another one. Third, interactions may occur between components in the sense that high expressions can further strengthen each other.

## Ethnomathematics

Sternberg's Investment Theory of Creativity can be used to gain general hints from a psychological perspective to promote creative action. For the domain of mathematics or mathematics teaching, there are, of course, also specific didactic considerations in this regard. For us, a promising approach seems to be the use of ethnomathematical thematic fields, which also address cultural aspects and contexts of mathematics. For students, these open up new topics and new ways of looking at these topics. That could not only lead to a better understanding of the focused concepts and procedures (Rosa & Orey, 2016) but could also encourage a creative examination of the issues.

## Mathematics from old times and foreign countries – workshops for primary students

In the fourth grade, towards the end of primary school age in most federal states of Germany, most students have already gained a lot of mathematical knowledge and considerable experiences in doing mathematics. Nevertheless, in regular classes frequently predominates an orientation on algorithms and pure numeracy skills to prepare students for secondary school. Especially in this phase of "student life", it seems important to us to strive for the learners' interest in mathematics and to enable them to gain experiences in the creative investigation of mathematically challenging situations. Therefore, we have designed the workshop series *"Mathematics from old times and foreign countries"*, which is aimed especially at fourth graders, but can also be attended by younger and older mathematics learners. In every multi-hour workshop the students work on a mathematical topic from a foreign country or a long-past era. In the first phase, they deal intensively with concrete materials; in the subsequent second phase, a detachment from the material and further investigations are processed on a more abstract level.

The intention of the workshops is to motivate the students to find out and to solve their own mathematical problems thinking about completely new and unknown situations. In these situations, the students are as explorers encouraged to develop new mathematical concepts and tools. The themes we developed origin in arithmetic, geometry and combinatorics. Working about only one of our themes as well as about a sequence of mathematical situations from old times and foreign countries, the students find out the mathematical richness of our culture by interesting examples.

In the following list, you shall find some thematic fields of the substantial learning environments (Wittmann, 1995) we developed:

- A first situation comes from Middle America. The Indians developed high symmetrical yarn polygons to adorn their tents. The students' task is to find out and to classify these symmetries (cf. the following section).

- The native place of the second topic is the north of Europe. Using very simple geometrical puzzle stones the students create their own polygon puzzle.
- The bone scratches of Africa give the starting point for an arithmetical journey to the world of numbers. The main idea is here the bundling.
- The Japanese abacus is devoted to number theory too. The task is to find out the commons and the differences between the well-known abacus of Adam Ries and the Asian abacus.
- The tracery od medieval monasteries in southern Europe motivates students to geometric investigations and to create their own rosettes.

Such a series of workshops can be particularly appropriate to implement suggestions from Sternberg's theory: According to our experience, the excitement and curiosity concerning the foreign is very high among most children of this age. Therefore, the embedding of mathematical questions in unfamiliar and thus often exciting historical and cultural contexts leads to a *high motivation* of students. As the above list exemplifies, ethnomathematical themes can be found, which, on the one hand, require little *domain-specific knowledge* and are thus easily accessible. On the other hand, they offer mathematical challenges for an investigative approach and make it also possible to gain heuristic experiences (focusing on strategies, as systematic testing, with high relevance for the primary school age). The teacher is responsible for creating an appreciative and supportive *environment*, which can be supported by the realization of the workshops at non-school learning places. In the workshops, the students always use different modes of representation. The example extensively presented in the following section shows that the thematic fields can also be explored from different mathematical perspectives (arithmetic, geometric, combinatorial ...) at different levels and with different goals. This addresses the learners' *abilities* of divergent thinking. Through a longer series of such workshops, the above-mentioned *thinking styles* and *personality* attributes may also have a positive effect.

In addition, due to the alienation of the students they may develop an attitude of an explorer. Relatedly, the students are more likely to experience themselves as creators, on the one hand, on the level of concrete products, which can be viewed from a mathematical perspective, and on the other hand on the level of theoretical ideas and concepts that allow a processing in a mathematical way.

## OJO DE DIOS – AN EXAMPLE

Ojos de dios, God’s eyes, originated from Huichol Indians in Mexico. They are made of crossed sticks (at least and very often two) and colourful yarn (cf. Fig. 1). Weaving an eye is seen as a meditative practice but you can also get your creativity flowing in developing very complex patterns. And you can do (mathematical) investigations, especially you could ask for special types or the number of different Ojos de dios. For instance, if you work with symmetric crosses of three or four sticks, you only mention the Ojos’ shape and identify symmetric ones, then there are 5 respectively 16 different Ojos. (cf. Fig. 3. For more sticks, we leave the question open for the reader.) From a mathematical point of view, the combination of combinatorial and geometric aspects for answering these questions seems very interesting for us.

## Some experiences with fourth-graders

A workshop on Ojos de dios for younger students can be organized in different ways. Based on our experience, we prefer an at least two-hour workshop. After a short introduction to the Ojos' origin and the handling of yarn and sticks, the students start their weaving on a cross of sticks which was prepared by the teacher at best. In workshops for fourth-graders it is recommended to work with crosses of two or three sticks. In the second part of the workshop, the students exchange the concrete materials for more abstract worksheets and were confronted with the "research question" of how many different Ojos de dios do exist. In the last part, the students discuss and systematize their often very interesting results

In the following, we will report some experiences on a workshop with mathematically interested fourth-graders attending a math circle at our university. Already during the weaving of the Ojos de dios, creative aspects were very often shown, for example in a variety of colour designs or different numbers of used sticks. Sometimes, the randomly occurred variations of the sticks' angle of intersection were picked up by the students; a girl even produced a three-dimensional Ojo de dios.


Fig. 1: Ojos de dios in different variants.

Many creative moments also developed during the subsequent more mathematical work on the Ojos de dios. First of all, when concretizing the "research question" the students intensively discussed when two Ojos should be considered as different: Should the colour or the size matter? Which features of the geometric shape should be considered? Finally, the students agreed to recognize only symmetric arrangements of sticks and to distinguish Ojos by the sticks used for weaving. In addition, rotationally and axially symmetric figures should be identified. Within this framework, for example, in the later discussion of the results the two Ojos in fig. 2 were considered as identical.

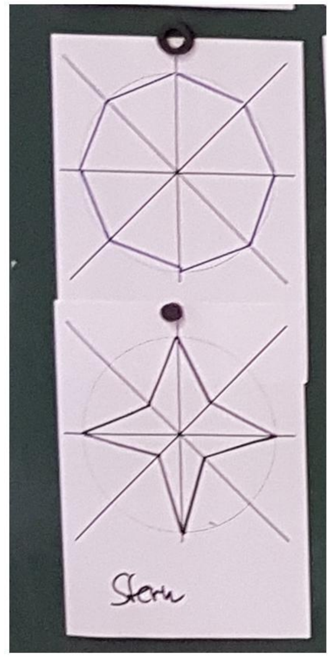


Fig. 2: "Octagon" and "Star" are the same.

How can you find as many different shapes as possible on crosses of two, three or four sticks? How do you know that you have found all possible shapes? In which (others convincing) system can the results be presented? How could found shapes be named? On all these questions the students found creative answers on their own. For example, the following figure shows all polygons on a cross of four sticks. They were all found by the students themselves. Following the students' suggestion, the polygons were arranged according to the number of vertices for the purpose of examining their completeness.

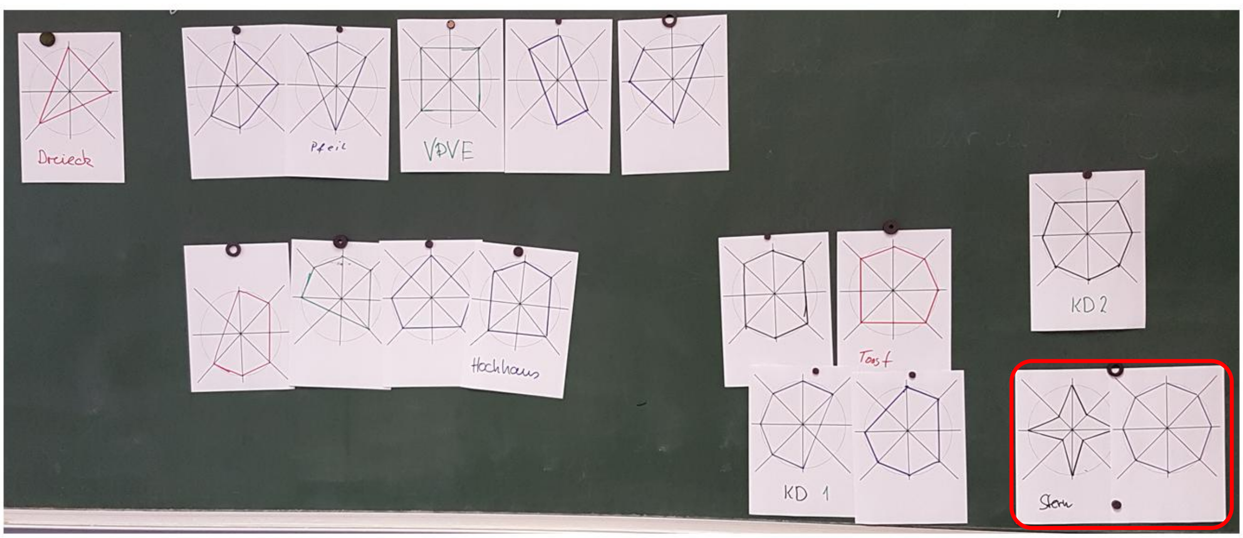


Fig. 3: Ojos de dios with four sticks, partly named by students.
(The marked Ojos are the same from the combinatorial perspective.)

Finally, after the presentation of the results some students could also formulate creative follow-up questions. For example, one student wanted to know the length of the strand of yarn used for his hexagonal Ojo de dios. This exemplarily makes evident that there are potentials for creative and mathematically interesting work, for which the use of concrete materials is necessary.

## FURTHER RESEARCH QUESTION

The report on the Ojos de dios has shown many potentials for fostering younger students' creative mathematical activities. In addition, such substantial learning environments also seem interesting to us from the perspective of mathematics education research. In initial exploratory studies, we are currently investigating the following questions:

- What potential do different materials (original materials, didactically prepared materials, more abstract worksheets ...) have for creative mathematical action?
- What other elements, such as student questions, student discussions or results, can ignite creative processes? Are there typical structures of communication or characteristics of student results regarding the initiation of creative processes?
- Are there different cross-situational types of creative processes in doing mathematics in primary school age? If so, how can these types be described?

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# DEVELOPING MATHEMATICAL CREATIVITY IN PRE-SCHOOL EDUCATION

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***Abstract.*** *Could creative problem solving be the object of work in pre-school education? This study followed the work of fifteen, four and five year old children and their teacher during a two month process of solving combinatorial problems with a large number of solutions. Findings show that all children responded positively to the problems, were successful in solving them and developed sophisticated strategies during the process.*

*Key words: problem solving, fluency, flexibility, originality, elaboration.*

## THEORETICAL FRAMEWORK

Guilford (1959, 1967) describes creativity as a dynamic mental process including both divergent and convergent thinking. He goes on to describe the four components for divergent thinking as fluency-the ability to generate multiple ideas for solving a problem, flexibility-the ability to generate a variety of ideas concerning a single problem, originality-the ability to generate novel ideas and, elaboration- the ability to describe, extend and develop an idea. Of course when talking about young children we do not expected of them to create something new and of significance to the entire society or its individuals. Generally speaking, it is almost impossible for young children to create anything new (Kudryavstev, 2011). What can be expected, in terms of creativity in young children, is for the children to rediscover mathematics and reproduce its essential features- thus acting as novice mathematicians.

There is a large body of literature linking mathematical creativity to problem solving - especially mathematical problems with multiple solutions (Chamberlin and Moon, 2005; Elwood, 2009; Plucker et al, 2004; Levav-Waynberg and Leikin, 2012). At the pre-school level such mathematical problems can be problems which can be approached in different ways (Shiakalli and Zacharos, 2012; Shiakalli et al 2015), as well as mathematical problems which have a number of solutions greater than one (Shiakalli and Zacharos, 2014).

Kaufman and Sternberg (2006) note that fostering creativity depends powerfully on the learning environments while Haylock (1987) proposes that it is the role of the teacher to identify, encourage and improve creative mathematical thinking at all levels of education. More recently, Neumann (2007) underlines the importance of an interactive learning environment in the development of mathematical creativity.

In this paper we present the work of fifteen pre-school children (aged 4-5½, of a rural public Cyprus Pre-School setting) while working on combinatory mathematical problems. By closely following their work we seek to answer the questions whether children as young as 4 and 5 years old are able to solve complex combinatorial mathematical problems with a large number of solutions.

## METHOD

This study formed a part of a broader educational programme, an action research teacher professional development programme, which extended throughout the school year (October 2014- June 2015). It included the development of educational activities aimed at creating investigative learning environments in mathematics through structured lesson sessions and "Free and Structured Play Time" (Free and Structured Play Time is daily from 07:45-09:05. During this time children are free to choose and participate in playful activities aiming at developing cognitive, social, emotional and kinetic skills and abilities).

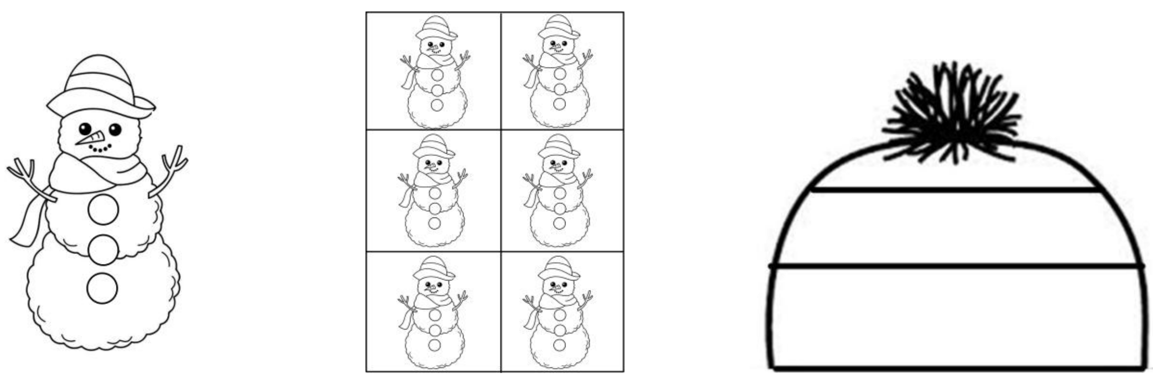
By developing the three combinatorial mathematical problems (presented in this paper) teacher and researcher anticipated that, while repeating the combinatorial problem solving process, the children would develop strategies for: (i) finding original solutions, (ii) developing fluency and flexibility in creating new solutions, and (iii) elaborating on discovered solutions in order to find new ones. Data was collected through a) videos of the teaching interventions and children's work at the Mathematics Table during "Free and Structured Play Time" (which were later analyzed by teacher and researcher based on an observation grid developed for the purposes of this study) , b) researcher field notes and c) teacher's reflective diary .

## THE MATHEMATICAL PROBLEMS

Children were asked to find all possible solutions to three combinatorial problems (the problems, manipulatives and graphical representation material are described in Table 1). The permutation without repetition problem was introduced first, and remained at the Mathematics Table during "Free and Structured Play Tine" for a period of three weeks. During this period children could chose to solve the problem as many times as they wanted (Table 2). In order for the problem to be solved, all six solutions had to be detected and graphically represented. After the permutation without repetition problem, the permutation with repetition problem was introduced. The teacher introduced the problem during Circle Time encouraging children to compare the two combinatorial problems, note similarities and differences and predict the number of solutions. The second permutation with repetition problem was set by the children: they were asked if they wanted to solve another combinatorial problem with even more solutions, were given the basic scenario "Snowy likes to colour hats" and through conversation with the teacher created the problem (Table 1). Again the children were encouraged to compare the three problems and this time (having had the second problem experience) estimate the number of solutions. The children worked on the two permutation with repetition problems in a similar way: a table was set for this task where up to four children could work simultaneously during "Free and Structured Play Time" containing coloured pencils and printed cards showing a snowman/hat (only one figure printed per card), a printed figure of a snowman / hat and buttons (in the case of the hat the buttons represented the colour each section of the hat would be coloured into). The children would use the manipulatives to create a solution. If the solution was original they would go on to graphically represent it (colour in a card) and place it on the wall. All original solutions were placed next to each other on the wall so that children could easily compare their solution to the ones already detected. The first permutation with repetition problem was solved within twelve days while the second permutation was given fifteen days. After each period, respectively, during Circle Time all solutions were placed at the centre of the circle and children were encouraged to talk about them and group them. Grouping the second problem solutions helped the children detect the solutions they had not found.

| Mathematical Problem Description | Manipulatives | Graphical Representation Material |
|---|---|---|
| *Permutation without repetition:* "Snowy has three buttons, each of a different colour. In which different ways can he place them on his tummy?" (6 solutions) | A drawing of Snowy (Figure 1a), a box of buttons | Answer sheet (Figure 1b), coloured pencils |
| *Permutation with repetition-1:* "Snowy found a box with red, green and yellow buttons. In which different ways can he place three buttons on his tummy?" (27 solutions) | A drawing of Snowy (Figure 1a), red, green and yellow buttons. | Answer cards (only one solution represented per card), coloured pencils |
| *Permutation with repetition-2:* "Snowy likes colouring hats. He has red, blue, green and yellow pencils. In which different ways can he colour his hats?" (64 solutions) | A drawing of the hat (Figure 1c), red, green, yellow, blue buttons (representing the colour of each hat section). | Answer cards (only one solution represented per card), coloured pencils |

Table 1: The mathematical problems, manipulatives and graphical representation material.



(a) (b) (c)

Figure 1: Mathematical problem manipulatives and answer sheet for the first problem.

**RESULTS**

All children chose to work on the initial permutation without repetition problem more than once (Table 2). During the repetition of the process children were observed to have used and developed different strategies for solving the problem.

| Subject and Age | Times problem was solved |
|---|---|
| S1(4), S4 (4,5), S5 (4,5), S9 (5,1), S15 (5,6) | 2 |
| S3(4,4), S8(5), S10(5,2), S12(5,4), S13 (5,4) | 3 |

| | |
|---|---|
| S7 (4,9), S11 (5,3) | 4 |
| S14 (5,5) | 5 |
| S2 (4,2), S6 (4,7), | 6 |

Table 2: Times each child chose to solve the first mathematical problem.

During their work all children were observed to had created and applied a strategy in order to solve the first combinatorial problem (Table 3). Children's work analysis showed that, not only were they able to develop and apply a strategy but they also refined it in future applications of the mathematical problem solving process.

| **Strategy** | **Strategy Description** | **Subjects** |
|---|---|---|
| *Detecting solutions randomly* | Placement of buttons in different places randomly and checking the answer sheet in order to avoid graphical representation of same solutions. | S1, S2, S3, S4, S5,S14 |
| Combination of deliberate alternation of colours and random placement | Placement of a different colour in fist button position and random placement of two remaining buttons in second and third position, for detection of three fist solutions. Random placement of buttons and checking answer sheet to avoid representation of same solutions- remaining sic solutions | S1, S2, S4, S5, S6, S8, S9, S10, S11, S12,S14, S15 |
| *Detection of solutions per noncontiguous pairs* | Placement of different colour in first place and random placement of colours in second and third place for identification of first three solutions. Return to a recorded solution placement of same colour in first place and reversal of colours in second and third places thus creating noncontiguous pairs of solutions. | S3, S6, S7, S8, S10, S11, S12, S13, S15 |
| *Detection of solutions per co contiguous pairs* | Detection of first solution with random placement of three colours. Detection of second solution with placement of the same colour in the first position and reversal of colours in second and third positions (creation of a pair of solutions). A change of colour in the first position and random placement of colours in second and third position. Then a reversal of colours in second and third positions (creation of second pair of solutions). Same process for the creation of the third pair of solutions. | S3, S6, S7,S10, S11, S13, S14 |

Table 3: Strategies developed during the initial problem solving process.

While working on the two permutations with repetition problems, the children showed to had developed persistence and patience as well as a positive attitude towards error. The refinement of strategies suggested elaboration while their ability to detect new solutions suggested fluency, flexibility and originality.

During the process of solving all three combinatorial problems the teacher would randomly come close to the children and encourage them to talk about their work, explain what they were doing , what they had already done and how they were planning to continue their work. When she saw that a child was finding difficulties in continuing the process (especially during the first problem solving process) she would sit next to them and work with them either by offering a helping idea or by posing helpful questions. When a child completed their work they would often call the teacher to show her their answer sheets. The teacher always reacted with enthusiasm and would express her surprise about the child's accomplishment. At the end of each process the teacher would ask the child how they felt, what they liked about the process and what they found difficult during the process.

**DISCUSSION**

In the present study we looked at pre-school children's ability to solve complex combinatorial problems with the use of graphical representations. Concerning our research question, whether pre-school children can successfully solve complex combinatorial mathematical problems with a large number of solutions, our findings show that young children apply the mathematical problem solving process in order to solve such complex mathematical problems. We think that different factors might have been influential. Firstly, the gradual development of the problems during a long period of time which enabled children to systematically work on the problem. Secondly, the teacher's contribution in (1) enabling and supporting the development of children's autonomy and (2) organizing the classroom in a way which gave the possibility for creative interaction to be developed. During the mathematical problem solving process, children demonstrated creative skills and abilities, such as fluency flexibility, originality, elaboration, persistence, patience and positive attitude towards error. Our findings also suggest that young children are able to use graphical representations in order to (a) detect original solutions, and (b) elaborate on existing solutions in order to detect all possible problem solutions.

Our findings are in accordance with other findings (Neumann, 2007; Kaufman and Sternberg, 2006; Haylock, 1987) underlining the important role of the teacher in setting and sustaining a creative environment. In our study, the classroom teacher played a central role in the development of a safe creative environment although throughout the process she was not the exclusive centre of the process. The children's expectations and interest were transferred to the experimental atmosphere created by the teaching situation, scaffolding their autonomy and providing multiple interaction experiences.

Lastly, in attempting to comment on the practical consequences of our findings, we could suggest that introducing young children to complicated mathematical problems within an environment of safety and encouragement could lead to the development of dynamic mental process including both, divergent and convergent thinking supporting them to rediscover mathematics acting as novice mathematicians.

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# QUESTIONS ABOUT IDENTIFYING TWICE EXCEPTIONAL STUDENTS IN A TALENT SEARCH PROCESS

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***Abstract.*** *A high potential in a certain (or in several) domain(s) does not necessarily mean a protection against developmental disorders or impairments. In many cases high potentials may mask the special needs of a child and, conversely, special needs may also mask a high potential. Because of the masking effect, even thoroughly executed talent search programs can fail to identify affected students. To learn more about the character and the amount of affected students a questionnaire was given to more than 400 parents whose children participated in a talent search program. The results show that some twice exceptional students can be identified, but the question remains whether identification in group processes is sufficient to give twice exceptional students the possibility to show as high potential as not affected students.*

*Key words: Twice exceptional, gifted, masking effect.*

## TWICE EXCEPTIONAL STUDENTS

"Twice exceptional students are students with high potential on the one hand and educational special needs on the other hand" (Nolte 2017). There are many approaches to define twice-exceptionality, which all have to deal with the problem that theoretical approaches to giftedness as well as to impairments and developmental disorders and learning disabilities differ broadly. So they can only be regarded as attempts to capture the problem.[1] Both have a dynamic and a systemic perspective on the development of competencies in common. Unfolding the potential of a child depends on the interplay of inborn factors, of intrapersonal factors and of environmental variables. Different models about giftedness like the model of (Gagné, 2004, 2005), the munich model of giftedness (Heller, 2000) and the actiotope model (Ziegler & Phillipson, 2012) underline the importance of the activities of a child under certain conditions.

With their model Betz & Breuninger (1982) and Breuninger (2014) describe the development of achievement in the context of learning disabilities and disturbances and also the interplay of several influencing factors. Their model exemplifies the importance of attributions of the interacting persons which lead to beliefs. Beliefs lead to activities which can be supportive or obstructive. Besides these aspects for twice exceptionality also information about the role of impairments and developmental disorders which can built barriers in learning processes are important. Compensating or overcoming barriers can be supported by methodical interventions so that impairments or developing disorders not necessarily lead to difficulties in learning processes.

The following model combines the above described models about giftedness and learning disabilities. Starting with learning preconditions including specific potential as shown on the left side, unfold themselves influenced by intrapersonal and environmental variables in

[1] For an overview of the actual discussions see Scherer et al. (2016); Singer et al. (2016).

learning processes, mathematical competencies. The arrows displayed this general learning process refer to the interplay of different factors. Thus, learning preconditions and previous knowledge can directly affect the motivation and endurance of a child. The experience of success also supports the self-efficacy of a child. But, also attitudes of teachers, parents and classmates towards the child are influenced by the success (or lack of success) of a child and therefore lead to certain interventions. Impairments or developmental disorders may build barriers that need to be compensated or overcome. Negative experiences and attributions may become a hindrance for unfolding a child's potential.

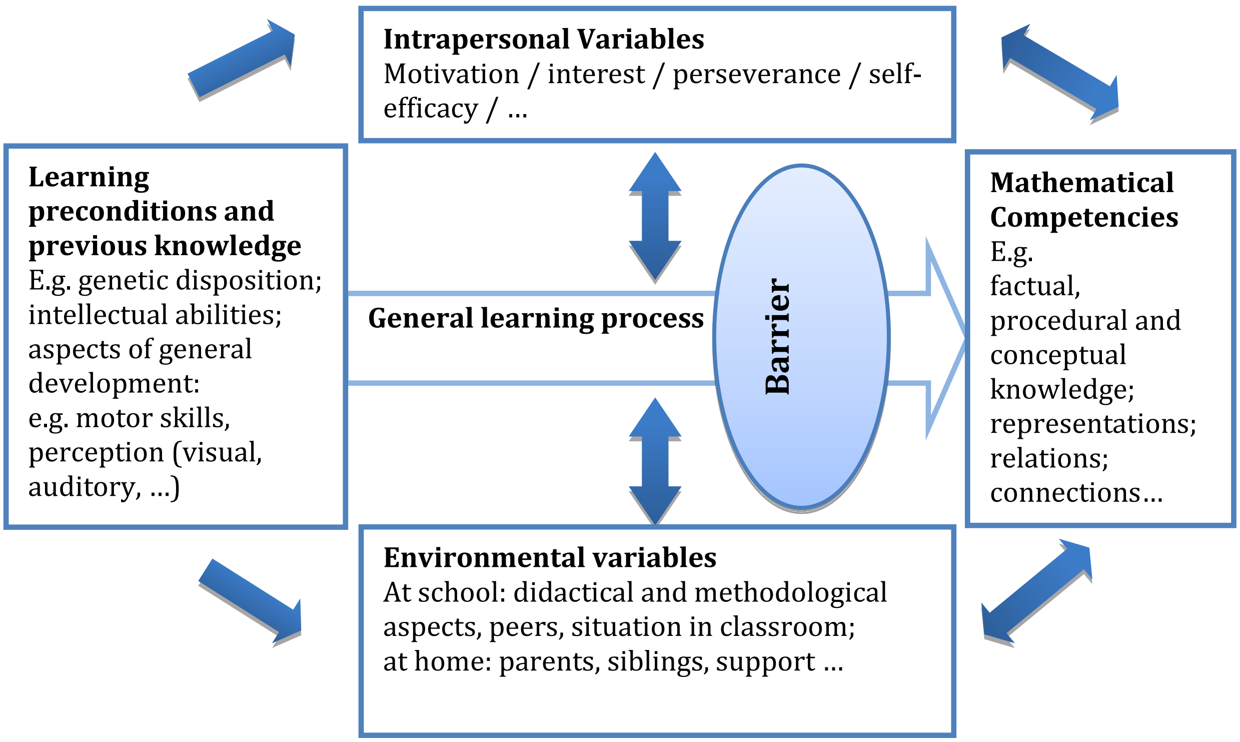


Figure 1 Influencing variables of learning processes (Nolte 2017)

On the teacher's side (environmental variables) significant for successful interventions are not only attributions, but also professional competences like pedagogical content knowledge (Shulman 1985). In order to be able to adequately assess the behavior and the achievement of twice exceptional children and to take appropriate methods, knowledge of different professions is needed. Thus, the noticing approach (see Sherin et al. 2008) is extended in order to take into account also appearances of impairments and developmental disorders. Not in all cases behavioral phenomena of developmental disorders are recognized as disorders because normally this is not part of the professional knowledge of a mathematics teacher.

## TALENT SEARCH PROCESS

Since school year 1999/2000, at the University of Hamburg we are fostering mathematically talented children starting with 8 years old children within the framework of the project called PriMa. This project is a research project and a project for fostering mathematically talented children.[2] To identify 50 promising students out of about 14,000 we offer a talent search process starting with trial lessons which are followed by a mathematics test and an intelligence test.

Especially at a young age the identification of a high mathematical potential is very demanding because the mathematical knowledge of children is still small. A search for talents poses the risk that children may be wrongly classified as especially talented or that children's talents are not recognised (see Heller, 2000). If there are impairments or developmental disorders the identification of a high potential is even more demanding.

## The masking effect

Twice exceptional students are at risk that impairments, disorders or learning disabilities, are not recognized. Often their high potential makes it possible to compensate barriers. Conversely, the impairments or developmental disorders hide a high potential which thus is not recognized. "When gifts and handicaps exist in one individual, they often mask each other so that the child may appear "average" or even as an "underachiever" (Silverman, 1989, p. 37). This phenomenon is called *masking effect* (Nolte 2017). In these cases, the barrier shown in the model prevents the development of high performance corresponding with their potential. A questionnaire we gave to parents of children who participate in our fostering project shows, that about 15% of the children were diagnosed as having developmental disorders (Nolte, 2012). In addition, some parents' information indicated further barriers, but the parents did not consider them as relevant to learning processes. During the fostering process we identified them as barriers for the students.

In order to reduce this masking effect already during the talent search process, a questionnaire was given to the parents of the participating children. Being informed about the children's problems offers the possibility for us to interpret their behaviour differently, e.g. if a child avoids writing answers or needs more possibility for movements than other children. The questionnaire contained questions about impairments and developmental disorders diagnosed by a physician, but also questions about frequent diseases during childhood and therapeutic interventions. These questions should also point to children with mild disorders whose relevance in challenging learning processes is not fully understood by all parents.[3] Out of about 400 parents, 132 returned the questionnaire.

Altogether 41 children got a diagnose.[4] Out of these we invited 7 children to participate in the fostering project. Under the category "others" parents mentioned distractibility and inattention, which belong to traits of ADHD. Therefore, in the first column in brackets, the total number of children with either ADHD or these disorders is noted, and in the last column the total number of children with other disorders.

[2] PriMa is a cooperation project of the *Hamburger Behörde für Schule und Berufsbildung* (u.a. *BbB*), and the *William-Stern* Society (Hamburg), the University of Hamburg.

[3] Working as a therapist for children with dyscalculia, I learned that many children with partial function disturbances got a treatment before they entered school (e.g. occupational therapy).

[4] Some children got several diagnoses.

| Our child got the diagnoses | | | | | | |
|---|---|---|---|---|---|---|
| ADHD | Dyslexia | Auditory pd | Visual pd | ADD | Emotional difficulty | Others |
| 2 (8) | 7 | 8 | - | 5 | 3 | 27 (19) |

Table 1: Distribution of diagnoses (pd = perception difficulties)

No child was mentioned having weaknesses in visual perception. This can be expected because in mathematical learning processes, visualizations are used to exemplify contents. In problem solving processes visualizations are also an important tool.

The parents' statements are contradictory. Thus, 103 parents indicated that no diagnoses were available, but 15 of them reported about therapeutic intervention, which is financed in Germany only after a corresponding diagnosis.

Not all children whose parents answered the questionnaire, participated in the complete talent search process. Among the 21 children who did this and were not identified as mathematically gifted, the performance during the talent search process was not sufficient in comparison with the results of other children. Ranking 5 and 6 in the mathematics test shows performance at a level which is too low. But it is questionable whether the tests really correspond with the potential of a twice exceptional child if compensatory possibilities were used to overcome the barriers. In another study (Nolte, 2012), the results of the number sequence test (NST) correlated strongly with the results of the mathematical test. This did not happen with child 1, 3, 4, 5, 7 and 8.

Conversely, with child 6, the results of the number sequence test (NST) do not match the results of the mathematical test. The results of some children, e.g. child 2 suggest an isolated mathematical ability. Child 10 has insufficient German language knowledge. This could explain the discrepancy between the number sequence test (NST) and the other results.

| 1. | IQ | IQ (lexicon) | IQ (NST) | Sum Maths Test | Ranking MT | Ranking TL | Diagnose |
|---|---|---|---|---|---|---|---|
| 2. | 130 | 127 | <145 | 29 | 5 | 2 | ADD |
| 3. | 110 | 87 | 139 | 46 | 3 | 2 | ADD |
| 4. | 93 | 112 | 133 | 5 | 6 | 2 | Audit. pd |
| 5. | 112 | 115 | 132 | 27 | 5 | 2 | Audit. pd + Dyslexia |
| 6. | 124 | 109 | 133 | 16 | 6 | 3 | ADHD |
| 7. | 118 | 111 | 85 | 52 | 2 | 3 | ADD |
| 8. | 104 | 130 | 90 | 16 | 6 | 3 | Audit. pd + Dyslexia + emotional d |
| 9. | 147 | 109 | 141 | 28,5 | 5 | 4 | ADD |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10. | 110 | 100 | 90 | 1,5 | 6 | 4 | Audit. pd |
| 11. | 103 | 81 | 141 | 9,5 | 6 | 3 | Speech therapy, Mig |

Table 2: Not invited children: Results of the different tests: IQ NST (number sequence test), MT (mathematics test), TL (trial lessons), audit. pd (auditorial perception difficulties), d (difficulties), mig (migration background)

The following table shows the results of some of the invited children. Children with just sufficient results in the mathematical test could compensate them with the results of their intelligence tests.

| | IQ | | IQ (NST) | Sum Maths Test | Ran-king | Ranking | Diagnoses |
|---|---|---|---|---|---|---|---|
| | 145 | 112 | 145 | 105,5 | 1 | 2 | X |
| 2. | 151 | 121 | 139 | 77,5 | 1 | 2 | X |
| 3. | 134 | 124 | 133 | 73,5 | 1 | 1 | Motoric d |
| 4. | 128 | 124 | 129 | 66 | 2 | 1 | X |
| 5. | 136 | 130 | 130 | 63 | 2 | 1 | X |
| 6. | 147 | 106 | 141 | 61 | 2 | 2 | gifted |
| 7. | 153 | 109 | 117 | 51,5 | 2 | 3 | ADHD |
| 8. | 146 | 115 | 130 | 46,5 | 3 | 3 | Problems because of late identification of being left handed |
| 9. | 145 | 133 | <145 | 46,5 | 3 | 3 | X speech therapy |
| 10. | 131 | 106 | 138 | 45 | 3 | 3 | X speech therapy, mig |
| 11. | 126 | 121 | 115 | 54 | 2 | 2 | X speech therapy |
| 12. | 130 | 100 | 117 | 38,5 | 4 | 3 | Motoric d |
| 13. | 138 | 124 | 120 | 30,5 | 4 | 3 | Audit. d |
| 14. | 136 | 124 | 123 | 29,5 | 4 | 2 | X |

Table 3: Invited children: Results of the different tests: X (no diagnose), IQ NST (number sequence test), MT (mathematics test), TL (trial lessons), audit. pd (auditorial perception difficulties), d (difficulties), mig (migration background)

The questionnaire also offered the possibility of free responses, which are about the support of the school. The statements of parents about the support in school are very different. Many feel supported, others report that the teacher does not recognize the necessity of support for a child to overcome a barrier. Many parents report that their children get exhausted by coping with the normal learning conditions in school or are bored or deny participating in lessons.

## DISCUSSION

The results of the questionnaire show that some of the twice exceptional children are able to successfully participate in the talent search, but not all of them. Although we paid special attention to twice exceptional children, we cannot provide reliable information about their potential. This is a general problem in talent search processes, but it is particularly difficult in the context of twice exceptionality because of the masking effect. Barriers result from impairments and developmental disorders, if this is not addressed by parents and teachers. Open answers from the parents underline the need for professional development of teachers so that they are able to recognize the relevance of disorders to learning processes, to develop appropriate methods and, if necessary, to seek for interdisciplinary collaboration with experts.

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# "MATHE MIT PFIFF" – A PROJECT AIMING AT EXTRACURRICULAR ENRICHMENT AT SCHOOL

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***Abstract.*** *In Germany, programs aiming at the support of children who have high mathematical interests and potentials are – irrespective of contests and specific schools – mainly conducted at several universities. Usually, they combine research, the education of student teachers and the support of children. An example is given by the long-term enrichment project "Mathe für kleine Asse" at the University of Münster. By contrast, reports on similar programs conducted at German schools are rare; conceptual exchanges seem to be rather difficult. In this article, the conception of the project "Mathe mit Pfiff" will be presented aiming at a transfer of established cornerstones of "Mathe für kleine Asse" into schools by extracurricular courses.*

*Keywords: mathematical giftedness; support program; enrichment*

## INTRODUCTION

In Germany, irrespective of recommendations of tasks suited to foster mathematical interests and potentials in regular classrooms (e.g., Nolte & Pamperien, 2016), specific and established support programs mainly can be found in the form of enrichment courses organized at several universities, mathematical contests, or both special "STEM"-classes and schools. Additionally, mathematical working groups are organized at many schools, but usually their conceptions are not published; thus, neither their organization can be reproduced, nor their sustainability can be assessed. The following questions crop up: Which organizational cornerstones are important? How can these cornerstones be conceptualized and retained, so that they can easily be transferred into different schools? One approach to find answers to these questions is to adapt well proven designs of university programs. In this article, we focus on a practical example of transferring structures of a university support program into a regular school: "Mathe für kleine Asse" ("math for small pundits") is a long-term enrichment project at the University of Münster founded by Friedhelm Käpnick in 2004; it consists of different groups between the 1st and the 8th grade (see for details: Benölken, 2015). While its sessions take place at university and its concept is imbedded into the education of student teachers, a transfer of its established organizational cornerstones into a school in form of an extracurricular course was focused on by the sub-project "Mathe mit Pfiff" ("math with a kick"), which was conducted between 2010 and 2016 at the Anne-Frank-Gymnasium in the City of Werne. The aim of this article is to summarize and to discuss the project's conception regarding its constitutive theoretical framework, its purposes, its applied diagnostic procedures, the organization of its sessions, and notes about the conception's evaluation from both children's and teachers' perspectives.

## THEORETICAL FRAMEWORK OF MATHEMATICAL GIFTEDNESS – AN OVERVIEW

As to giftedness, a scientific consensus on the following aspects can be assumed (e.g., iPEGE, 2009): The phenomenon is described as a potential for visible performance; it demands the consideration of cognitive and co-cognitive as well as intra- and interpersonal determinants, and it occurs domain-specifically. With regard to mathematics, particular criteria have been emphasized such as remembering and structuring mathematical facts, sensitivity and fantasy, transferring structures, intermodal transfer and reversing thoughts (Käpnick, 1998). Overall, giftedness has to be seen as a dynamic phenomenon that demands a holistic view on individual personalities, complex long-term process-diagnostics and support as early as possible. The mentioned aspects are synthesized within the approach of Fritzlar, Rodeck and Käpnick (2006), who constructed a model that describes the development of mathematical giftedness (see also the equivalence as to primary school age: Fuchs & Käpnick, 2009). It provides the base of diagnostic procedures of "Mathe für kleine Asse" and of the sub-project "Mathe mit Pfiff": Summarized, mathematical giftedness is seen as an above-average potential, in particular as to the criteria mentioned above, characterized by individual determinants and a dynamic development depending on inter- and intrapersonal influences in interdependence with supporting personality traits: high mental activity, inquisitiveness, willingness of efforts, joy at task solving, ability to concentrate, insistence, self-dependence or skills of cooperation, to name but a few (cf. Benölken, 2015). Even if a transfer of a university program to a school-environment will entail a more heterogeneous field of participants, the model seems to provide an adequate framework of courses organized at schools as well, in particular since intrapersonal aspects like motivation are considered. Thus, an open approach is given to elect children showing high cognitive potentials, strong mathematical interests or the like (similar to the term of "mathematical promising", cf. Sheffield, 2003).

## PURPOSES OF THE PROJECT

"Mathe für kleine Asse" is imbedded into the education of teachers at the University of Münster; there are three dimensions of purposes focusing (1) on the development of student teachers' professional knowledge, (2) on the support of children, and (3) on widespread research in the field of giftedness (for details: Benölken, 2015). "Mathe mit Pfiff" provides one of several extracurricular activities at the Anne-Frank-Gymnasium in different domains like chemistry or languages (for details see: www.afg-werne.de). Against this background, the aims of "Mathe für kleine Asse" were adapted, but reduced to the following aspects: (1) In view of the children, their joy at task solving and their intellectual inquisitiveness should be fostered, and they should gather an adequate imagination of mathematics and activities of mathematicians (similar to the university groups). (2) As to research, the main purposes were the transfer of the conception of "Mathe für kleine Asse" to regular schools and its evaluation.

## PROCESSES OF DIAGNOSTICS AND APPLIED TOOLS

Corresponding to the outlined view above on mathematical giftedness, diagnostics within the project "Mathe für kleine Asse" are organized as a long-term process consisting of four steps: First, teachers of schools in Münster suggest children as to a participation in the project reflecting typical criteria of giftedness; second, a first meeting in order to give the children the opportunity to experience the project's organization and atmosphere at the university; third, an introductory test; fourth, long-term process-diagnostics during each child's participation synthesizing different standardized and non-standardized tools (for details: Benölken, 2015). Altogether, the diagnostic procedures are quite complex according to the strongly selected group of children from different schools in the City of Münster. Against the background of transferring the concept into a school, the procedures were partly adapted, but their complexity was reduced: Since it had to be expected that within the participants of "Mathe mit Pfiff" there would be fewer children who have very high potentials than in the university groups, more children were likely to be attracted by joy at doing mathematics or the like. Additionally, diagnostic procedures which were similarly complex to the university groups seemed to be unsuited in comparison with the school's extracurricular courses in other domains. As a consequence, diagnostics were organized in three steps: In a first step, the teachers elected (similarly to the university groups) children considering both typical criteria of giftedness and aspects of a child's motivation; hence, a widespread view was taken into account. In a second step, the elected children together with their parents decided whether or not they wanted to take part in the project. In a third step, process diagnostics began applying nothing more but observations and specific rating sheets (Fuchs & Käpnick, 2009) – their collection provided a basis for consultations with teachers and parents.

## ASPECTS OF PROJECT SESSIONS' ORGANIZATION

Even if children of the 5th and the 6th grade participated in the very first school year, the project mainly aimed at supporting children of the 5th grade. They usually met every two weeks after regular lessons between the autumn holidays and the end of the school year, i.e., about 15 times per school year. In each of the university courses, as many as 30 children take part. They are guided by scientists and not more than ten student teachers per group (according to the aim of their education). Therefore, 90-minutes children's sessions are guided by preparing and reflecting workshops, e.g., to discuss notes of every child's mathematical potential (for details: Benölken, 2015). By contrast, "Mathe mit Pfiff" was conducted without student teachers, but by the author as a guiding scientist and a mathematics teacher of the school who is interested in the field of mathematical giftedness, and who participated in some advanced trainings. Every school year, about 20 children took part in 90-minute-sessions. There were no preparing or reflecting workshops. Instead, based on an informal exchange of diagnostic impressions of each child's characteristics and developments, the teacher and the author composed reports (maximum one page) at the end of a school year. In this way, a base of consultations or feedback was given, mainly taking into account facets as to both motivation and criteria of mathematical giftedness.

Beyond special-issue-sessions like mathematical excursions or testing sessions, complex task fields are the main form of methodological organization in the university groups (examples are given by: Fritzlar, Rodeck, & Käpnick, 2006). Due to the reduced capacities, only the last type of organization was conducted within "Mathe mit Pfiff". Such a complex task field covers one theme per 90-minutes session and it should consider the idea of natural differentiation as much as possible by these aspects: (1) It should attract children's inquisitiveness and joy at problem solving. (2) Its mathematical substance should be adequate and possible discoveries should cover a large range from rather superficial observations to real deepness. (3) It should be very open as to different working materials, solutions' styles, ways or presentations. (4) It should provide possibilities to create problems that are attracted to the given (see also: Benölken, 2015). A session is divided into three parts: First, a problem is presented that leads to the core of the theme within about 10 minutes. Second, children turn into research activities for about 60 minutes organizing themselves as to social modes of working, applied working materials, ways of describing their solutions or the like. At the end, the children discuss their solutions, while particular attention is given to the quality of argumentations or reasoning. An example of task fields is given by the theme "numbers of diagonals in polygons" (German translation from “Anzahlen von Diagonalen in Vielecken”; Käpnick & Benölken, 2015). In the first stage, the children are asked how many diagonals maximally can appear in a four-sided figure. Thus, a first example is focused on to discuss terms like “diagonals”, and maybe hypotheses or possible strategies. Finally, the main research question follows: How many diagonals can appear in arbitrary polygons maximally? In the second stage, children turn to research activities organizing themselves as to, e.g., social working modes, working materials (geoboards, fold paper, sticks or the like) or forms of solutions’ presentations. In the third stage, ideas, results and argumentations are discussed. Usually, there is a wide range of deepness between relatively simple observations emerging from “handmade” approaches and complex solutions using tables or sequences of pictures to decline patterns or a formula (examples are shown by Figure 1).

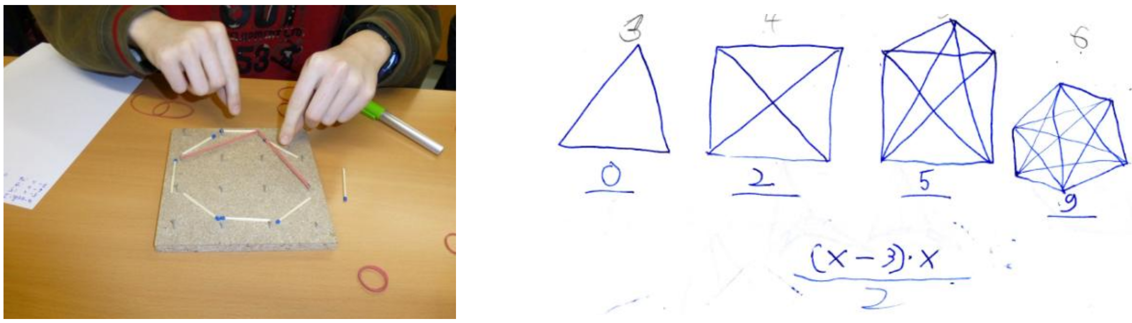


Figure 1: Impressions of children’s approaches and solutions.

## NOTES ABOUT THE CONCEPTION’S EVALUATION

The conception of “Mathe mit Pfiff” was evaluated by a first small study focusing on teachers’ perceptions on the one hand, and, in particular, by a long-term study focusing on children’s perspectives on the other hand (cf. Benölken, 2011).

The *aim* of the first study focused on teachers’ observations about possible impacts of taking part in the program as to children’s acting in regular mathematics lessons. An explorative *design* was advisable, since individual perceptions in a closed school-context were focused, even if, as to the *method*, a simple standardized questionnaire was applied. It consists of statements which reflect some impacts of the participation that seemed to be most important in the consensus view of the author and the guiding teacher (for example: “The children show a higher willingness of efforts.”), and which was assessed by a three-step-scale between “-“, “0” and “+”. The *sample* contains the teachers of the first conduction of the program in the very first school year (N=6). As to the *procedure*, the teachers fulfilled the questionnaires at the end of the school year at home – basic information about the purpose was given in the head. The *analysis* was conducted by descriptive statistics. Overall, the *results* shown in Table 1 indicate that impacts of participating in the program were observed, and their characteristics obviously seem to be positive, which might be interpreted as a first indicator of the program’s sustainability (cf. Benölken, 2011).

| | - | 0 | + |
|---|---|---|---|
| The children’s feedback about the program is positive. | 0 | 1 | 5 |
| The children suggest tasks applied in the program in order to implement them into regular math lessons. | 1 | 2 | 3 |

| | | | |
|---|---|---|---|
| The children's self-confidence has become more advantageous. | 0 | 2 | 4 |
| The children's motivation has increased. | 0 | 1 | 5 |
| The children show a higher willingness of efforts. | 0 | 1 | 5 |
| The children's joy at task solving has increased. | 0 | 1 | 5 |
| The children's social competencies have improved. | 2 | 4 | 0 |
| The children's competencies as to arguing and reasoning have improved. | 0 | 1 | 5 |
| I observed negative changes as to the children's acting. | 4 | 2 | 0 |
| The children show overconfidence. | 6 | 0 | 0 |

Table 1: Absolute frequencies of teachers' answers on impacts of the participation.

The *aim* of the second study was to assess children's motives regarding their participation. Similar to the first study, an explorative *design* was advisable. As to the *method*, a standardized questionnaire which has been applied in the project "Mathe für kleine Asse" for many years (Fuchs & Käpnick, 2009) was adapted. It consists of closed assessments about intrinsic (e.g., "I took part in the project, since I love solving puzzles.") and extrinsic motives (e.g., "I took part in the project, since my parents want me to.") regarding the participation, which simply could be marked by a cross (multiple answers possible). The *sample* contains the program's participants between 2010 and 2016 (N=112; 43 girls and 69 boys). The questionnaires were fulfilled at the end of the participation. All *procedures* of questioning were consistent. As to the *analysis*, descriptive statistics were applied. Overall, the *results* show that the children took part mostly due to intrinsic motives, and, thus, a high level of sustainability regarding the program's conception is indicated (Table 2). In addition to the results shown by Table 2, gender-specific differences were not observed.

| | I took part in the project, since ... | abs. (rel.) frequency |
|---|---|---|
| intrinsic motives | ... I love solving puzzles. | 87 (approx. 78%) |
| | ... I am often bored in math lessons. | 46 (approx 41%) |
| | ... math is my favourite activity. | 79 (approx. 71%) |
| | ... I want to know more about math. | 77 (approx. 69%) |
| | ... I am very good in math. | 96 (approx. 86%) |
| | ... I enjoy difficult mathematical tasks. | 81 (approx. 72%) |
| | ... I want to know something about math skills of other children. | 27 (approx. 24%) |
| extrinsic motives | ... my parents want me to. | 18 (approx. 16%) |
| | … my teacher wants me to. | 12 (approx. 11%) |
| | ... my friends want me to. | 1 (approx. 1%) |

Table 2: Absolute and relative frequencies of children's answers on their motives to participate.

## CONCLUSION

Overall, the evaluation studies might be interpreted as first indications that the transfer of the conception of "Mathe für kleine Asse" to an extracurricular course at school in the design of "Mathe mit Pfiff" might be a practicable approach to organize support programs for mathematically gifted students at school. Of course, the studies are nothing more but explorative, even if they show simple possibilities of assessment from different perspectives. However, a comprehensive evaluation would have to cover more aspects such as the question how students' ability to solve problem tasks might change because of their participation. The transfer presented in this article raises some exemplary key questions that could be considered when organizing similar projects at schools: (a) Which theoretical knowledge of mathematical giftedness is essential for the guiding teacher? (b) Which age group will be focused? (c) How could the sessions be organized? (d) What kind of tasks could be suitably applied? (d) Is the intention of the program barely focusing on children's support, or should diagnostic procedures be implemented, too? – and if, which instruments might be suited? (e) How could the program be evaluated from different views? Even if the presented conception offers some ideas or exemplary answers to these questions, from a research perspective, the question crops up how such answers could be given in a different manner; and following up on this, major challenges stem from empirical investigations on approaches or elements that might prove their value from a larger view.

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# THE LOCAL MATHEMATICAL CONTEST AS AN ASSEMENT TOOL OF STUDENTS' MATHEMATICAL CAPACITY

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***Abstract.*** *The paper discusses the mathematical circle at the Correspondence School of Mathematics for primary school students. The circle aims to introduce mathematically interested students to the first steps of problem solving. The composition principles of local mathematical contest are presented. The rationales for evaluating the students' attainments in the contest are discussed. The collected data characterise the levels of students' performance of certain problem-solving skills and suggest guidelines for further running of the mathematical circle.*

*Key words: Mathematical contest, mathematical circle, problem solving.*

## INTRODUCTION

The *Correspondence School of Mathematics* (CSM) at the University of Latvia organizes mathematical Olympiads (Regional, State Olympiad and Open Mathematical Olympiad) and offers extracurricular contests for students of various age. Experts' evaluation of students' contest works shows that a considerable part of these works demonstrates substantial incompleteness of specific mathematical knowledge necessary to solve the tasks. The problems offered at the contests considerably differ from the regular mathematics curriculum. Students' insufficient experience in problem solving points to the need for training and preparation to successfully participate at the Olympiads.

Not all of the young students have an opportunity to increase their mathematical knowledge in extracurricular activities. CSM noticed high activity and interest in mathematics from young contest participants. For example, the youngest group of Open Mathematical Olympiad accounts for more than a fifth of the more than 3000 participants during the last years (any student from the 1$^{st}$ to 5$^{th}$ grade may participate in the youngest group; mostly these are 5$^{th}$ graders). This interest is one of reasons why CSM started a *mathematical circle* (MC) especially for young students. With reference to Piaget's theory (1964) of cognitive development, students from 4$^{th}$ and 5$^{th}$ grade gradually cross the threshold form the concrete operational stage to the formal operational stage. Extracurricular mathematical activities can help them develop correct mathematical concepts and acquire specific heuristics of problem solving.

## GOALS OF THE MATHEMATICAL CIRCLE

The importance of math circles is recognised worldwide. Mast (2015), for example, underlines that mathematical circles must be accessible to all students, not simply to students who excel in mathematics. She notices that this environment may appeal to students who do not do well in math classes at school. They can discover the dynamic and creative nature of mathematics.

The mathematical circle for beginners started at CSM in September 2016. Any student of 4$^{th}$ or 5$^{th}$ grade can participate in the classes until the login is available on the homepage of CSM. The classes take a place at the University on Friday evenings for circa an hour. The main goal of the mathematical circle is to introduce students to the first steps of *problem solving* (PS) process:

- understanding of the formulated task,
- investigation of given objects,
- introduction to heuristic PS methods,
- formation of mathematical language,
- evaluation of the solution and of the answer.

Different topics are discussed in the classes to reach these goals: questions on number theory; combinatory technics to classify the objects; visual methods to solve linear equations; problems to develop spatial imagination; and other topics. Every meeting occurs in a friendly atmosphere with an open discussion space. The class starts with some introductory tasks, then a challenge is proposed that students have to research individually and share their findings. The teacher gives explanations and introduces the methods for solving the proposed challenge.

## WHY THE LOCAL MATHEMATICAL CONTEST IS COMMENDABLE?

Local mathematical contests on MC are organized regularly. Mathematical circles of Moscow Center of Continuos Mathematics Education[1] include written works one or two times per year. Monthly contests take place at the Berkeley Math Circle[2]. Math Circles at the University of Waterloo provide the semesters with final contests[3].

The local December contest in MC at Correspondence School of Mathematics was carried out for two reasons: to offer an opportunity to the students to test their knowledge themselves in the spirit of competition that is an advanced motivation to work more persistently; for teacher to collect data for evaluation of the development of students' mathematical skills and their progress in acquiring the methods examined in the classes.

The evaluation of students' works is an important aspect of the contest. The researchers in mathematics education present different models of the problem-solving process and elaborated frameworks for evaluation of PS processes, mathematical thinking, and mathematical competencies. The team of TIMSS defined the mathematical competencies as a list of such general skills as mathematical thinking, argumentation, problem posing and solving, and other skills (*MEASURING STUDENT KNOWLEDGE AND SKILLS,* 1999). They created a detailed assessment framework for 4$^{th}$ graders devoting 50% of tasks to the students' competencies with numbers, including some pre-algebra concepts (Mullis & Martin, 2013). The creators of NRICH Project identified mathematical thinking strategies needed to tackle problems and formed the basic model of PS process (Piggot, 2011). To identify the key points of students' performance in problem solving, Yevdokimov and Passmore (2008) distinguish four qualitative characteristics of PS: the first step of PS; main

[1] http://www.mccme.ru/circles/mccme/2017/ homepage of MC on Moscow Center of Continuos Mathematics Education
[2] http://mathcircle.berkeley.edu/monthly-contest homepage of the Berkeley MC
[3] http://www.cemc.uwaterloo.ca/events/mathcircle_presentations_gr6.html homepage of the Waterloo MC

information extracted from a problem; generalisation possibly required for solution of a problem; completion of the solution.

## EXAMPLES OF PROBLEMS AT THE DECEMBER CONTEST

The December contest was prepared to evaluate only certain mathematical competencies of students useful for problem solving: understanding of problem, use of mathematical terms, combinatory and algorithmic abilities, immersion in the problem solving, skills of justification, and spatial imagination.

The contest included 9 tasks: a couple of warm-up tasks and problems from algebra, number theory, combinatory theory, combinatorial geometry and algorithms that students had to solve during an hour. They had to write answers and explain some solutions. The contents of the contest differed a little for 4th and 5th grades.

A rebus was included to detect how deep the student can immerse in the problem – does he or she find multiple solutions:

*Problem 2 (Grade 5).* Some of digits in the equation are replaced by " * ". Find the erased digits to get a correct expression! $7* - 6* = *9$

The solution of Diophantine equation needs argumentation based on the divisibility of numbers:

*Problem 4 (Grade 4).* For which natural numbers N and M is the equation true?

$2 \cdot M + 10 \cdot N = 121$

The following problem was presented to the 5th graders to show their comprehension of decimal numbers:

*Problem 4 (Grade 5).* For which digits ( * ) is the equation true? $A* + *A = AA$

Students can test their systematisation skills with problem 5:

*Problem 5 (Grades 4 and 5).* Arthur wants to pick 3 toys from 7 marble beads, 2 whistlers, and 4 dolls. How many choices does he have?

The problems 6 and 8 were inspired by the problem set of Russian mathematical Olympiads (Agahanov et al., 2002). Problem 6 for 4th grade has 8 different solutions. To solve this problem, students must do arithmetical calculations and check their answers. A similar problem for 5th graders needs the proof of negation.

*Problem 6 (Grade 4).* Arrange the numbers 1, 2, 3, 4, 5, 6 in the magic triangle (see figure 1.a))!

*Problem 6 (Grade 5).* Can you arrange four different natural numbers in the magic triangle for the sum in all triangles to be equal (see figure 1.b))?

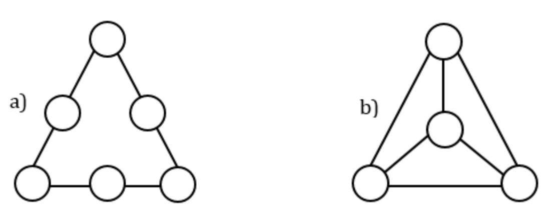

Figure 1 : Magic triangles for a) Grade 4; b) Grade 5

To solve problems 7 and 8, students have to apply spatial imagination and must justify their solution. Problem 7 is a typical contest problem. With problem 8, students need to write the algorithm for the illumination of storage, and they have to explain why some aisles will stay in dark.

*Problem 7 (Grades 4 and 5).* Colour some unit squares of the square 6 x 6 for every rectangle 3 x 1 in this square to contain at least one coloured unit square! What is the minimal number of coloured unit squares?

*Problem 8 (Grade 5).* The map shows the aisles of a storage (see figure 2). The dots mark the buttons of light switches. If someone pushes the button, he switches the light on in all aisles connected with this point if they were dark before or he switches the light off in the opposite case. How many aisles is it possible to light up at the same time if at the starting moment the storage is in the dark?

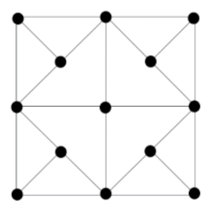

Figure 2 : The map of aisles and light switch buttons of storage

Solutions of some tasks did not give the relevant information for qualitative analysis of students' works. These were warm–up tasks (arithmetical calculations in mind) and the last problem about the detection of winning strategy for a simple two-player colouring game. No-one solved this problem. Only a few students remembered the argument of symmetry but did not use it correctly.

## STUDENTS' PERFORMANCE

There were 24 participants from 4th grade and 15 participants from 5th grade at the December contest. For every problem, the level of performance of a specific competence applied in the solution was evaluated. Students presented rather short solutions, therefore they were classified in three levels: high level if student presented a qualitative answer, middle level if the problem was solved partly, low level if the problem was not solved correctly.

Table 1 shows the number of students who solved the appropriate problem and the number of students who solved the problem partly. This elementary statistic demonstrates that many contestants had misunderstandings, made mistakes or did not solve some problems at all. Participants did not enter the problems thoroughly, as soon as they found the first suitable answer. They did not work systematically and they did not search for the additional properties of the given objects. Some of the students did not understand the requirements formulated in the problems. To solve the simple combinatory problem 5, contestants had to create a list of combinations of three elements. A significant part of them did not understand this problem completely and therefore gave only a few examples.

| Total number of students | Problems | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 4th grade: | Solved | 12 | 20 | 21 | 3 | 3 | 0 | 1 | 4 | 0 |
| (24 participants) | Partly | 12 | 0 | 0 | 12 | 16 | 9 | 9 | 16 | 19 |
| 5th grade: | Solved | 8 | 3 | 1 | 3 | 4 | 8 | 8 | 4 | 1 |
| (15 participants) | Partly | 7 | 12 | 11 | 6 | 6 | 5 | 2 | 10 | 13 |

Table 1: The summary of results of solved problems

Table 2 shows the levels of students' performance related to a specific competence. Most students had problems with the use of mathematical language – they did not correctly express their thoughts or did not use definite mathematical terms. It made the argumentation and justification less qualitative. No one student gave the explanation of the estimation of extreme values asked in problems 7 and 8. Only five 5th graders added arithmetical calculations as an argument of minimal coloring in the solution of problem 7. Spatial imagination was a weak point of the youngest students –problem 7 was too abstract for them. No one of the 4th graders immersed deeper in the problems – they stopped as one possible solution was found. The oldest contestants applied more scrutiny – some of them presented a complete answer to the problem with multiple solutions. In summary, there were 4 works among 24 participants from 4th grade with middle and high performance level of the mentioned competencies and 6 works among 15 participants from 5th grade that were evaluated highly as well.

| Performance/ Competencies | Termi-nology | Systemati-sation | Spatial imagination | Justification | Immersion | Algorithm |
|---|---|---|---|---|---|---|
| 5th grade | | | | | | |
| High | 3 | 7 | 8 | 5 | 3 | 6 |
| Middle | 6 | 2 | 2 | 5 | 3 | 6 |
| Low | 6 | 6 | 5 | 5 | 9 | 3 |
| 4th grade | | | | | | |
| High | 5 | 8 | 1 | 6 | - | 2 |
| Middle | 7 | 9 | 10 | 9 | - | 8 |
| Low | 12 | 7 | 13 | 9 | 24 | 14 |

Table 2: Number of students by performance level of competences

## DISCUSSION

The collected data show the difference of problem-solving skills between the participants of MC of 4th and of 5th grade. Approximately a third of the group of oldest students presented good competencies in problem solving, whereas the proportion of high performers in the youngest group was less. The lower performers account for approximately half of the 4th graders. These results indicate diverse levels of primary mathematical knowledge of the participants of MC. Only half of youngest students agreed in a short questionnaire that they have good results in school mathematics. They commented that they did not understand some problem-solving methods in detail. The 4th grade students cannot always give their full attention to the problems discussed in classes. They need more tasks with concrete actions like drawing, combining geometrical figures, or playing mathematical games.

Summary results of the December contest point to the need to pay attention to the understanding of the offered problems and discussing what is given; what are the conditions; what is unknown. Students' argumentation mode has to be developed. Students have to apply the evaluation of gained results.

In order not to lower the students' self-esteem it is important to provide challenging and accessible problems. Students' attitude toward mathematics plays an important role in the learning process. For example, Teylor (1992) presents some interviews of mathematics life history to research the complex nature of a persons' attitude that is interconnected with thinking, acting and feeling. The teacher has to organize MC classes so as to increase students' interest in mathematics. In his experimental studies Vygotsky (1978) revealed the significant role of the teacher as a child's mentor in the problem solving process. The guidance of a skilled teacher helps the student to bridge the actual deficient knowledge with the attainable goals.

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# PEDAGOGICAL PRACTICES THAT FOSTER MATHEMATICAL CREATIVITY AT TERTIARY-LEVEL PROOF-BASED COURSES


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***Abstract.*** *Many mathematics education publications highlight the importance of fostering students' mathematical creativity in the tertiary classroom, however few describe explicit instructional methodologies to accomplish this task. Using Sriraman's (2005) five principles for maximizing creativity as a framework, we highlight three case studies for which pedagogical practices occurred to foster creativity. All three proof-based courses used a meta-cognitive tool named the Creativity-in-Progress Rubric (CPR) on Proving. This tool was developed to encourage students to engage in practices that research literature, mathematicians, and students themselves suggest may promote creativity in processes of proving. Centered around the CPR on Proving, we report on the variety of tasks, assignments, and in-class discussions the instructors used in their proof-based courses and verify that these did highlight mathematical creativity using student interview data.*



*Key words: proving, tertiary-level, teaching practices, meta-cognition*


## INTRODUCTION

Mathematical creativity is discussed as an important aspect in tertiary mathematics (e.g., Zazkis & Holton, 2009; Schumacher & Siegel, 2015), but pedagogical actions that support fostering creativity in classrooms are rarely mentioned or explicitly studied. Ervynck (1991) stated, "[W]e therefore see mathematical creativity, so totally neglected in current tertiary mathematics courses, as a worthy focus of more attention in the teaching of advanced mathematics in the future" (p. 53). In this paper, we share results of our preliminary exploration of this issue by attempting to address the research question: ***What teacher actions or practices in the proof-based tertiary classroom might foster students' perceptions of mathematical creativity?***

To explicate mathematical creativity with tertiary students, three instructors from different universities in the U.S. implemented various practices such as designing assignments, creating tasks, and structuring class discussions in their courses. One common feature of these practices was that all three instructors centered their implementations around a tool created to enhance research-based actions for mathematical creativity in the proving process, the Creativity-in-Progress Rubric (CPR) on Proving (Savic et al., 2016; Karakok et al., 2016). This rubric was developed considering certain theoretical aspects of mathematical creativity, which is discussed in the following section.

## THEORETICAL PERSPECTIVE AND BACKGROUND LITERATURE

There are over 100 different definitions of mathematical creativity (Mann, 2006) and multiple theoretical perspectives (Kozbelt, Beghetto, & Runco, 2010). In developing the CPR on Proving, the authors considered mathematical creativity as a *process* that involves different modes of thinking (Balka, 1974) rather than looking at the creative end-product (Runco & Jaeger, 2012). Mathematical creativity in the classroom often considers a *relative perspective* instead of *absolute creativity*. This means creativity is a process of offering new solutions or insights that are unexpected for the student, with respect to his/her mathematics background or the problems s/he has seen before (Liljedahl & Sriraman, 2006). Finally, many researchers (e.g., Baer, 1998) stressed the importance of domain-specific creativity: "creativity is not only domain-specific, but that it is necessary to define specific ability differences within domains" (Plucker & Zabelina, 2009, p. 6).

## Creativity-in-Progress Rubric (CPR) on Proving

The CPR on Proving was rigorously constructed through triangulating research-based rubrics (Rhodes, 2010; Leikin, 2009), existing theoretical frameworks and studies (Silver, 1997), studies exploring mathematicians' and students' views on mathematical creativity (Tang et al., 2015), and investigating students' proving attempts (Savic et al., 2016).

There are two categories of actions that may help a student foster mathematical creativity: *Making Connections* and *Taking Risks*[1]. Making Connections is defined as the ability to connect the proving task with definitions, theorems, multiple representations, or examples from the current course that a student is in or possible prior course experiences. Taking Risks is defined as the ability to actively attempt a proof, demonstrate flexibility in using multiple approaches or techniques, posing questions about reasoning within the attempts, and evaluating those attempts. Making connections has three subcategories (between definitions/theorems, between examples, and between representations), and Taking Risks has four subcategories (tools and tricks, flexibility, posing questions, and evaluation of a proof attempt) that are designed to have students explicitly think about ways to develop aspects of their own mathematically creative processes. Each subcategory has three benchmarks: beginning, developing, and advancing, and each benchmark is explained for the student or instructor to achieve.

## Teaching for Development of Creativity

The literature for teacher actions that develop mathematical creativity at the tertiary level is scarce. Zazkis and Holton (2009) cite a few implicit instances or strategies for encouraging mathematical creativity, including learner-generated examples (Watson & Mason, 2005) and counterexamples (Koichu, 2008), multiple solutions/proofs (Leikin, 2007), and changing parameters of a mathematical situation (Brown & Walter, 1983). From the pre-tertiary literature, there are several articles of fostering mathematical creativity through problem posing (e.g., Silver, 1997) or open-ended problems (e.g., Kwon, Park, & Park, 2006).

Sriraman (2005) conjectured five principles for maximizing creativity in a K-12 classroom: a) Gestalt, b) aesthetic, c) free market, d) scholarly, and e) uncertainty. While there has been literature that utilized these conjectured principles (e.g., Haavold, 2013), there has not yet been literature describing how one would implement such principles. The three case studies, even though limited to proof-based courses in a tertiary setting, may give examples of such implementation. Each principle will be accompanied by a short

[1] For a final version of the CPR on Proving, see Karakok et al. (2016).

description, a pedagogical action by one of the three instructors, and student data supporting that the action indeed linked to creativity.

## METHODS

### Participants

Three instructors, Drs. Eme, X and Omar, from three different institutions (two located in Western US and one located in Northeastern US) participated in this study. Each instructor used the CPR in Proving; Drs. Eme and X removed the word "creativity" from the rubric to minimize the explicit influence of the rubric on students' development of ideas on mathematical creativity. They also both implemented IBL teaching pedagogies, whereas Dr. Omar also included lectures. Dr. Eme introduced the CPR on Proving mid-semester in her Transition to Advanced Mathematics course in Spring 2016. Dr. X's implementation was in a seminar on Elementary Number Theory during Fall 2015 where he introduced the rubric in the third week. Dr. Omar implemented the CPR in his Combinatorics course in Spring 2016 and used it both homework and portfolio assignments.

Prior to the start of the semester, all instructors discussed their course goals with the researchers and shared their CPR on Proving implementation plans with the researchers. Drs. Eme and X were involved in the development of the CPR on Proving where as Dr. Omar approached the authors to utilize the CPR on Proving in a course that he was designing. All three instructors met with the authors regularly to discuss the process of their utilization of the CPR on Proving throughout the semester. All three instructors collected their students' work and utilization of the CPR on tasks. Dr. Eme also audio-recorded in-class sessions. Students in the three courses were invited to participate in interviews at the end of the semester. In this preliminary report, we share our preliminary coding of notes from instructors' self-reported actions in class and implementation plans, along with recorded implementations of the CPR in their courses. We intend to give "existence proofs" of the five principles described by Sriraman (2005) in a creativity-emphasized classroom.

## RESULTS

### Gestalt Principle

The Gestalt principle, according to Sriraman (2005), is associated with preparation or suitable engagement with a problem, an incubation or break period, and a subsequent insight or AHA! moment which is verified. Sriraman stated that students get opportunities to "engage in suitably challenging problems over a protracted time period, thereby creating the opportunities for the discovery of an insight and to experience the euphoria of the "Aha!" moment" (p. 26). Dr. Omar utilized the CPR on Proving while handing out problems labelled as "portfolio problems," which are, quoting from the syllabus, "much more involved, and the intention is to allow freedom to roam with it in any direction you wish." The students were required to use the rubric in a minimum three-page write up summarizing the proving processes they used. Unbeknownst to the students, many of these portfolio problems were open in mathematics and the one portfolio problem had the same weight as the other three problems in the assignment which Dr. Omar viewed as "exercises". The assignments were approximately two weeks in length. In an interview, Student 2 solved a problem using the Gestalt principle: "And eventually – I think I was actually falling asleep. As I was falling asleep I had a 'Eureka' moment of the problem. I figured out some important part of it."

## Aesthetic Principle

Sriraman (2005) stated that "mathematicians have often reported the aesthetic appeal of creating a "beautiful" theorem" (p. 27) The aesthetic principle applies to a teacher valuing solutions that utilize unusual proving techniques, come from diverse topics of mathematics, or make efficient or elegant solutions. While not explicitly valued, a product of the socio-mathematical norms created in Dr. Eme's class was that the students associated mathematical creativity with efficiency. This was by the CPR on Proving having a category labelled *Tools and Tricks*, and many students in her course noticing one student's efficiency in his proof by utilizing a tool or trick. Student P stated that it, "made the proof way smaller, way shorter, and it was interesting to see how he came up with that… [the trick] made the proof as efficient as it was. I think that is very creative."

## Free Market Principle

"Teachers should encourage students to take risks" (Sriraman, 2005, p. 28). The free market principle involves creating a classroom environment that allows students to freely input ideas, thoughts, and solutions. Two months into the course, Dr. Eme started a class period showing the class their own exam solutions: "...That's the exam 2 'solutions' and I say solutions in quotes because they're not all 100% correct, okay, but it doesn't matter. You know there are still really good ideas in there and that's what I want you to see." Her pedagogical action of explicitly valuing good ideas over correctness fostered an environment where students were encouraged to present new ideas.

## Scholarly Principle

Sriraman (2005) stated that the scholarly principle is creating a classroom environment "in which students are encouraged to debate and question the validity of… approaches to problems…, be encouraged to generalize the problem and/or the solution, as well as pose a class of analogous problems" (p. 28). This enables students to create and be participants at a scholarly level. Dr. Omar, in his course, posed some open problems in the field of combinatorics, so some of his students ended up in some absolute creative moments, and hence contributed to the scholarly field of combinatorics. Dr. Eme encouraged debate about problem approaches utilizing the CPR on Proving. The discussion below ensues:

1 Dr. Eme: What did you guys get for the first one?
2 Stephanie: Advancing
3 Dr. Eme: Advancing? Why?
4 Stephanie: Because they were able to utilize multiple theorems and definitions…Definition Q, the consecutive integers, Definition test 3.
5 Dr. Eme: Good. Good. Other people agree? Disagree?
6 Tony: Agree.
7 Dr. Eme: Agree? Ok. How about "between representations?"
8 Cargo: That "The Between Representations" still confuses because I'm not sure exactly what it means? Is it supposed to be like using the notation or what?

9 Dr. Eme: Yeah, that's a good question. Does anybody have an answer?

10 Stephanie: I would say it's anyway you can rewrite it, or draw a picture, or anything 13 you can do to represent that same concept but in a different way.

## Uncertainty Principle

The uncertainty principle "requires that students be exposed to the uncertainty and the difficulty of creating mathematics" (Sriraman, 2005, p. 28). According to Sriraman's conjecture, the teacher must take actions "cultivating this trait of perseverance" (p. 28). Dr. X asked the students to use the CPR on five occasions throughout the semester, during homework as an evaluative tool, and during the final exam as extra credit. For example, in a homework problem, a student provided some scratch-work in his proof, bracketed the scratch-work, and wrote the reason why this scratch-work was not leading to a correct proof (in the student's words, "a mistake"). The student then promptly proved the theorem, utilizing the evaluation subcategory of the CPR on Proving.

## CONCLUSION

This preliminary work into actions that maximize creativity in the tertiary proof-based classroom may be of assistance to present or future instructors. Specifically, this work can provide an entry point for instructors who value creativity yet are unsure about tools needed to foster creativity in their classroom. Each instructor had a different approach to fostering creativity, but each valued creativity, which we believe is an important first step. The CPR on Proving also seemed to facilitate the explicit valuing of meta-cognitive practice, which can contribute to creativity. As Katz and Stupel (2015) stated, "Creative actions might benefit from meta-cognitive skills and vice versa, regarding the knowledge of one's own cognition and the regulation of the creative process" (p. 69).

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# DOES INTELLIGENCE AFFECT ALL STUDENTS' MATHEMATICAL CREATIVITY?

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***Abstract.*** *The present study aims to investigate whether there is a relationship between intelligence and mathematical creativity and if it does so, in which manner this relation is differentiated between students of different degree of intelligence. Therefore, 476 students attending 4th, 5th and 6th grades of elementary school participated in this study, by completing two tests: one measuring their mathematical creativity and the other measuring their fluid intelligence. Confirmatory factor analysis verified the existence of a moderate relationship between intelligence and mathematical creativity. In particular, intelligence is a factor that might predict an individual's mathematical creativity. Furthermore, correlation analysis indicated that such a relation exists only between students of average intelligence, whereas in students with high or low intelligence, mathematical creative ability and intelligence behaved as independent variables.*

*Key words: mathematical creativity, intelligence, threshold theory of intelligence.*

## INTRODUCTION AND THEORETICAL CONSIDERATIONS

Research interest on examining whether there is a relationship between creativity and intelligence appeared early. As Runco (2007) mentioned, this interest can be explained by the researchers' effort to provide evidences that the creative ability is independent from other fields of psychology, such as intelligence. However, the results are still contradictory (Sternberg & O' Hara, 1999). The present study aims to investigate whether the two constructs, that of mathematical creativity and intelligence, are related and to examine if this relationship is differentiated between students of different degree of intelligence.

### Connecting mathematical creativity and intelligence

As the research results on the connection between creativity and intelligence are conflicting (Sternberg & O' Hara, 1999), the theoretical framework is organized as follow: we will firstly present research works that revealed that the two concepts are distinct and independent from each other and then we will review research works that support the existence of a relationship between them.

Among the first research attempts which revealed that creativity and intelligence have nothing in common was the one conducted by Getzels and Jackson (1962). These researchers worked with two groups of students. The first group consisted of students with high intelligence and low creativity, whereas the second group consisted of students with high creativity and low intelligence. The results of their study concluded that the two groups of students had not any statistically significant difference, except for their interests. Similar results were found by Wallach and Kogan (1965), who concluded that the correlation index between a creativity and an intelligence test was low. Silvia's meta-analysis (2008) on Wallach and Kogan's data (1965) repeated that the two concepts have no relationship between them. Kim (2008) also found a non-significant relationship between creativity and intelligence and thus he proposed that students with low intelligence might be creative, and vice versa.

On the contrary, several researchers considered that the relationship between creativity and intelligence is obvious and thus they investigated in which manner the two concepts are interrelated. For instance, in the Structure of Intellect model, that was proposed by Guilford (1967), creativity is an aspect of intelligence, in contrast to Sternberg and Lubart's theory (1996) in which intelligence is a necessary condition to emerge creative behavior. Nevertheless, the Threshold theory of intelligence (Torrance, 1962), combines the two tensions by suggesting that although creativity and intelligence are discrete and independent aspects they have a relation. In particular, a relationship between creativity and intelligence seems to appear in individuals with IQ score lower than 120, whereas in individuals with IQ score more than 120 the relationship between creativity and intelligence is negligible (Runco, 2007; Sternberg & O'Hara, 1999). Jauk, Benedek, Dunst and Neubauer (2013) also concluded with similar results.

As it is obvious from the abovementioned discussion, there is an absence on research studies that investigate the relationship between intelligence and creativity in mathematics. Therefore, this study will provide some empirical evidences to this direction. Specifically, this study aims: (a) to investigate whether intelligence and mathematical creativity are related; (b) to examine in which manner this relation is differentiated between students of different degree of intelligence.

## METHODOLOGY

### Data collection

For the purposes of the present study, 476 students of 4th, 5th and 6th grades (9-12 years old) participated, by completing two tests: a mathematical creativity test (MCT) and a fluid intelligence test (NNAT). In particular, the MCT consisted of four multiple solutions mathematical tasks. Participants were asked to provide as many, different and originals answers as they could, in 40 minutes. The evaluation of the tasks followed the assessment method proposed by Kattou et al. (2013): (a) Fluency score: we calculated the ratio between the number of correct mathematical solutions that the student provided, to the maximum number of correct mathematical solutions provided by a student in the population under investigation. (b) Flexibility score: we calculated the ratio between the number of different types of correct solutions that the student provided, to the maximum number of different types of solutions provided by a student in the population under investigation. (c) Originality score: it was calculated according to the frequency of a student's solutions in relation to the solutions provided by all the students. Therefore, three different scores yielded for each student in each task. The final score of the test was obtained by adding the respective scores of fluency, flexibility and originality in the four tasks and then by converging them to a scale ranging from 0 to 1.

Fluid intelligence was measured using the Naglieri Nonverbal Ability Test (Naglieri, 1997). This test includes 38 tasks that should be completed in 30 minutes. For each correct answer students were awarded 1 point, while wrong answers received 0 points. Both tests were examined for their reliability: the internal consistency of scores measured by

Cronbach's alpha was 0.783 for the MCT and 0.831 for the NNAT, which are regarded as satisfied (Murphy & Davidshofer, 2001).

### Data analysis

Data were quantitatively analysed using the statistical packages SPSS and the statistical modeling program Mplus (Muthén & Muthén, 1998). Firstly, we used SPSS to examine whether there is a correlation between mathematical creativity and intelligence, as it was measured with the abovementioned instruments. In the case that there was a correlation between the two constructs we would apply Confirmatory Factor Analysis using the Mplus, in order to examine whether an a-priori theoretical model - which assumes that intelligence predict mathematical creativity - can be empirically grounded for the given population. Afterwards, a correlation analysis took place, in an effort to investigate the existence of such relationship in students with different degree of intelligence.

## RESULTS

The results have been organized according to the aims of the present study.

### The type of relationship between mathematical creativity and intelligence

With regard to the first aim of the study, we conducted correlation analysis. The results of this analysis indicated moderate correlation indices among intelligence as measured by NNAT and the three components of mathematical creativity, namely fluency ($r = .365$, $p<.05$), flexibility ($r = .337$, $p <.05$), and originality ($r = .295$, $p <.05$). Although the correlations are not strong they are statistically significant. Then a confirmatory factor analysis took place in order to investigate the validity of an a-priori theoretical model. In this model we assumed that intelligence contribute to the prediction of an individual's mathematical creativity. For the evaluation of model fit, we took into consideration three indices: The chi-square to its degree of freedom ratio ($\chi^2$/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). According to Marcoulides and Schumacker (1996), an acceptable model should have the value of CFI higher than 0.90, the value of $\chi^2$/df lower than 2 and the value of RMSEA lower than 0.08.

The results of this analysis revealed that the proposed model matched the data set of the present study and determined the "goodness of fit" of the factor model (CFI=0.989, $\chi^2$=85.045, df=51, $\chi^2$/df= 1.667, RMSEA=.051). The structure of the model, as well as the corresponding loadings of the model are presented in Figure 1. As is presented in Figure 1, mathematical creativity is a second order factor that is made up of three first order factors, namely fluency ($r = .93$, $R^2 = .87$), flexibility ($r = .98$, $R^2 = .94$) and originality ($r = .97$, $R^2 = .95$), as it is proposed by Torrance's definition. The first-order factors are formed by the respective performances of students in the Mathematical Creativity Test (e.g. the factor Fluency is formed by the variables Fluency in Task 1, Fluency in Task 2 e.t.c.). Furthermore, between the three creative abilities which correspond to the same task (e.g., Fluency in Task 1, Flexibility in Task 1, Originality in Task 1), statistically significant correlations exist, ranging from .23 (Fluency 3 - Originality 3) to .76 (Flexibility 3 - Originality 3). The structure of the proposed model also addresses the hypothesis that students' fluid intelligence ($r = .50$, $p < .05$) is a predictor factor of mathematical creativity.

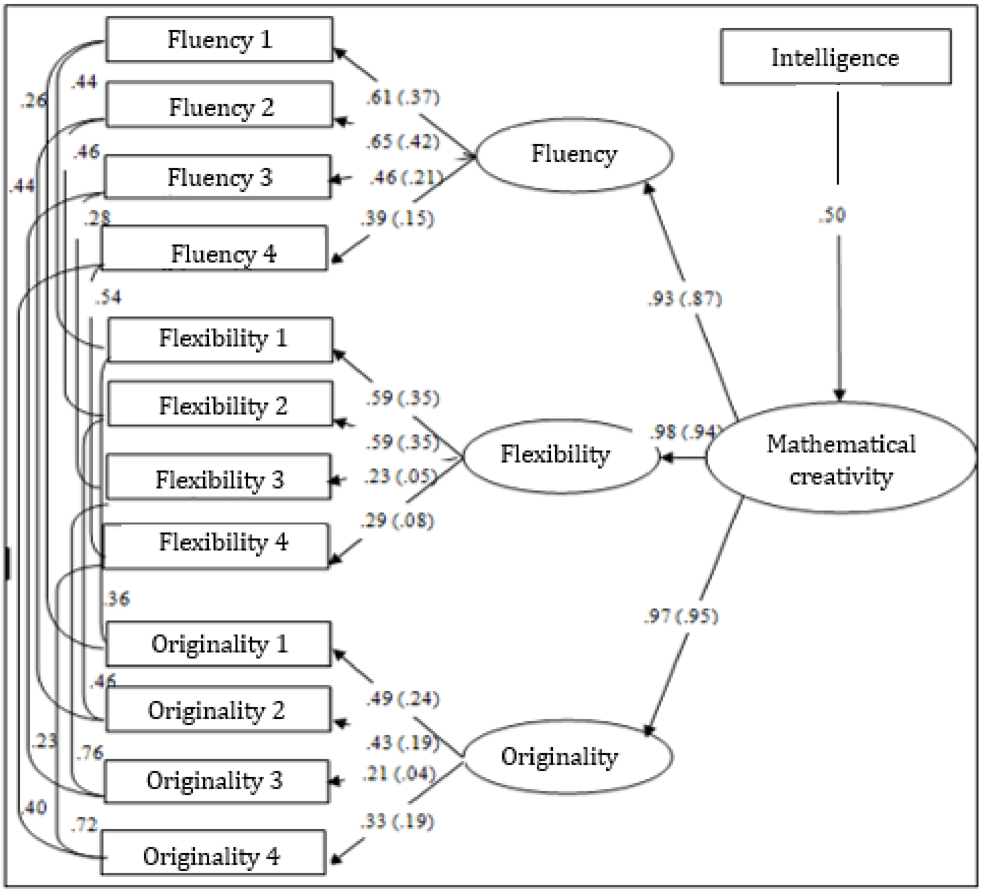


Figure 1: The structure and the loadings of the verified model.

## The relationship between mathematical creativity and intelligence in different groups of students

With regard to the second aim of the study we took into consideration the Threshold Theory of Intelligence and we attempted to investigate whether there is a relationship between mathematical creativity and intelligence in three groups of students, which vary according to their intelligence score in the NNAT. In particular, students in Group 1 had the lowest performances on the NNAT (lowest 5% of the scores in the NNAT). Group 3 comprised by students who received the highest score on the NNAT (higher 5% of the scores in the NNAT), whereas Group 2 comprised by students with average score in the NNAT. Table 1 presents the descriptive characteristics of the three groups of students.

| | Group 1 (N=18) | Group 2 (N=429) | Group 3 (N=29) |
|---|---|---|---|
| | $\bar{X}$ (SD) | $\bar{X}$ (SD) | $\bar{X}$ (SD) |
| Fluency | .318 (.121) | .363 (.129) | .475 (.136) |
| Flexibility | .247 (.104) | .273 (.081) | .348 (.091) |
| Originality | .342 (.187) | .432 (.168) | .571 (.156) |
| Intelligence score | 6.833 (1.581) | 18.655 (5.336) | 31.966 (1.569) |

Table 1: Means and standard deviations in the three groups of students.

Afterwards we conducted a correlation analysis between the elements of mathematical creativity (fluency, flexibility, originality) and intelligence, in the three groups of students. According to the results presented in Table 2, significant correlations between intelligence and creative ability in mathematics appeared only in Group 2, as opposed to Groups 1 and 3. In particular, in the population with average degree of intelligence (Group 2) the correlation coefficient between intelligence and fluency was r = .324 (p <.05), the correlation coefficient between intelligence and flexibility was r = .290 (p <.05) and the correlation coefficient between intelligence and originality was r = .220 (p <.05). However, for students who were in the upper 5% of intelligence test's performances (Group 3) mathematical creativity had no significant correlation with intelligence (p> .05). Similarly, there was no statistically significant correlation between intelligence and creativity(p> .05) in students with low IQ (Group 1).

| | Intelligence | | |
|---|---|---|---|
| | Group 1<br>r (p) | Group 2<br>r (p) | Group 3<br>r (p) |
| Fluency | -.206 (.411) | .324 (.001)* | .210 (.274) |
| Flexibility | -.043 (.865) | .290 (.001)* | .038 (.847) |
| Originality | -.154 (.542) | .220 (.001)* | .076 (.695) |

* Statistically significant correlation p<.05

Table 2: Correlation between intelligence and mathematical creativity in the groups of students.

## DISCUSSION

The aim of this study was to investigate the relationship between mathematical creativity and intelligence. Moreover, we examined whether the relationship between intelligence and mathematical creativity is differentiated between students with different IQ score.

A confirmatory factor analysis showed that mathematical creativity, as defined by the abilities of fluency, flexibility and originality, is correlated with intelligence measured by the NNAT. Specifically, intelligence can be a predictor of students' creative ability in mathematics. The above result is in agreement with Torrance (1962) who claimed that intelligence is a necessary, but not a sufficient condition for the emergence of creative behavior.

Moreover, the results of this study indicated that intelligence and mathematical creativity, measured with NNAT and MCT respectively, are related only to students with average IQ score. Furthermore, in individuals with high or low intelligence, intelligence and mathematical creativity are behaved as independent cognitive structures. This result seems to verify and extend the Threshold theory of intelligence (Runco, 2007; Torrance, 1962). As in the Threshold theory of intelligence, we detected a statistically significant correlation of mathematical creativity and intelligence in students with average degree of intelligence, as opposed to students with high IQ. In these students we did not find any statistically significant correlation, that might be an indication that there is a ceiling of intelligence; above from this ceiling there is not a linear relation between intelligence and mathematical creativity (Runco, 2007). Apart from this result, in the present study we found a low intelligence level; below of this level there is not any causal relationship between creativity and intelligence. In a similar conclusion was reached by Jauk et al. (2013), who observed a statistically significant relationship between intelligence and creativity in a range of intelligence scores; beyond this range the two entities are acting independently.

To sum up, the relationship between creativity and intelligence is not simple (Leikin, 2008). The relationship between the two concepts is differentiated among students with specific cognitive characteristics. Therefore, future studies could investigate the variation on mathematical creativity in students with different cognitive features, such as mathematical competence, memory and processing. At the same time, future work could examine in which way the interaction of cognitive characteristics with intelligence could affect students' creativity in mathematics.

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# SPATIAL ABILITIES AS PREDICTOR TO MATHEMATICS PERFORMANCE OF MATHEMATICS MOTIVATED STUDENTS

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***Abstract.*** *In the few years since the Kangaroo Contest arrived in Israel, we discovered that all winners in grades 2-6 had success in tasks oriented on spatial abilities (SA). In this study, we investigate a potential relationship between spatial abilities and mathematics performance (focused on non-standard problems) in mathematically motivated students (MMS) who participated in the Kangaroo Contest. We also checked whether the correlation between scores of SA tasks and the rest non-standard problems (RNSP) in the contest depends on the participants' age. Strong correlation between SA tasks and mathematics performance, as well as well-known malleable spatial abilities can lead us to the necessity of spatial abilities' development in early childhood as predictor of later mathematics achievement. This issue is important for students at all levels and especially for MMS whereas some of them will become later mathematically promising students.*

*Key words: spatial ability, mathematics performance, competitions, students' motivation*

## INTRODUCTION

In research literature, there is some evidence about a relation between spatial ability and mathematics performance. People who perform better on spatial tasks also perform better on tests of mathematical ability (Delgado & Prieto, 2004; Lubinski & Benbow, 1992; McLean & Hitch, 1999). This relation holds true throughout different ages (Gathercole & Pickering, 2000; Kyttälä, Aunio, Lehto, Van Luit, & Hautamaki, 2003), whereas greater spatial ability at age thirteen is associated with preference for mathematics-related subjects at age eighteen and helps predict success in STEM (Science, Technology, Engineering, and Mathematics) careers (Wai, Lubinski, & Benbow, 2009). Spatial ability at age 18 moderately correlates with raw SAT (Scholastic Assessment Test) mathematics scores, and remains a significant predictor of mathematical ability after controlling for general intelligence, processing speed and working memory (Rohde & Thompson, 2007). Some researchers even assumed that spatial processes are recruited for mathematics (Presmeg, 2006; Rasmussen & Bisanz, 2005).
The study we describe in this paper has begun to identify the relation between spatial ability and mathematics performance focused on solving non-standard problems in mathematically motivated students (grades 2-6) who participated in the Kangaroo Contest in Israel.

## SPATIAL ABILITIES

There are many definitions of spatial ability; it is generally thought to be related to skills involving the retrieval, retention and transformation of visual information in a spatial context (Velez, Silver & Tremaie, 2005) and includes the ability to manipulate the information represented in visual or graphical forms (Diezmann & Watters, 2000). Halpern (1986) explains that spatial ability is the ability to imagine what an irregular figure would look like if it is rotated in space. She adds that it is the ability to discern the relationship between shapes and objects.

In this paper, we utilize the following division of spatial abilities to components which are considered practical (Höffler, 2010):

*Spatial orientation*: The ability to perceive the positions of various objects in space, relative to each other and relative to the viewer, particularly across changes in orientation.

*Mental rotation*: Mental manipulation/rotation of remembered objects or elements in a scene.

*Spatial visualization*: Ability to perceive complex spatial patterns and comprehend imaginary movements in space.

## MATHEMATICS COMPETITIONS AS A MOTIVATING FACTOR

Mathematical competitions, in their recent form, have more than one hundred years of history and tradition. Kahane (1999) claimed that large popular competitions could reveal hidden aptitudes and talents and stimulate large numbers of children and young adults.

Robertson's study (2007) of the history and benefits of mathematical competitions reported on success in math competitions and in math achievement in general. It seemed to be linked to the love and interest instilled in students and an appreciation for math and problem solving methods, as well as the opportunity to acquire high-level skills with extra training and the development of a particular culture, which encourage hard work, learning, and achievements. Bicknell (2008) also found many advantages to be gained from the use of competitions in a mathematics program such as student satisfaction, the enhancement of students' self-directed learning skills, sense of autonomy and, co-operative team skills.

The interplay between cognitive, metacognitive, affective, and social factors merits particular attention by researchers because it may give us more insight into the development of mathematical potential in young learners (Applebaum, et al., 2013) .

Mathematical competitions are being organized in different forms, at different places and for different type of students. Being a part of challenging mathematics activities beyond the classroom, the model of the Kangaroo Contest offers to many students the opportunity to be exposed to aspects of mathematics, other than in a regular classroom. As such, it may help them to apply their skills to new situations and, at the same time, enrich their learning experience (Kenderov et al., 2009). We assume that mathematical completion, even one that does not target mathematically gifted, would attract such students; therefore, in investigating their abilities we could better understand their nature and how to foster their development in different cognitive domains. In this study, we investigate possible relationship between mathematics performance and spatial abilities in the context of Kangaroo tasks.

## KANGAROO CONTEST

Each year, over six and half million pupils in the ages 5-18, from more than 70 countries over the world, participate in the Kangaroo Contest.

The contest is composed of just one test: no selection, no preliminary round, and no final round. It takes place in March, on the same day and the same hour in all countries, and consists of twenty-four or thirty multiple choice questions of increasing difficulty. For each question, a choice of five answers (distractors) is provided.

The Kangaroo Contest is more of a game than an uncompromising competition (Dolinar, 2012). The most obvious difference is that the Kangaroo Contest is not just for the best mathematically talented students. Instead it aims to attract as many students as possible, with the purpose of showing them that mathematics can be interesting, useful and even fun. Though, sadly, it has generally become accepted that mathematics is difficult, very abstract and not approachable by the vast majority of people, the number of contestants in the Mathematical Kangaroo proves that this need not be the case. With a large number of competitors, the Kangaroo Contest helps eradicate such prejudice towards mathematics.
Choosing appropriate challenging tasks is an important condition in the successful contribution of mathematical competitions to the developing students' learning potential (Bicknell, 2008). In the case of the Kangaroo Contest, the problems are selected every year from a long list of problems provided by the leaders of all countries (more than 70) participating in the Kangaroo Contest. Contradictory to other competitions, the Kangaroo Contest problems are more appropriate according to the challenging task concept suggested by Leikin (2009). Such tasks should be neither too easy nor too difficult to motivate students to task completion and develop mathematical curiosity and interest in the subject.

**The Structure of the Kangaroo Contest**

The test consists of 24 multiple choice problems for grades 2-4 (and 30 multiple choice problems for grades 5-6). All the problems are divided to three groups. Each one consisted of 8 problems (10 problems for grades 5-6) and are rated according to their level of difficulty: problems 1-8 (1-10 for grades 5-6) are defined as Easy level, problems 9-16 (11-20 for grades 5-6) are defined as Average level, and the problems 17-24 (21-30 for grades 5-6) are defined as High level. Participants in the Kangaroo Contest have 75 minutes to cope with the problems. Using any accessories other than pens and paper is forbidden.
Each of the 24 problems (30 for grades 5-6) has five distractors; only one of them is the correct answer. The students are tested in different places all over the country and their tests are sent for evaluation to country leaders. All tasks in the Kangaroo Contest are different than the tasks students meet in their text books.

***The research questions***

In this study, we investigate the next questions:

1. What is the relationship between spatial abilities and mathematics performance (focused on non-standard problems) in mathematically motivated students (MMS) who participated in the Kangaroo contest?
2. Does spatial ability depend on participants' age?
3. Does the correlation between scores of SA tasks and the rest non-standard problems in the contest depend on the participants' age?

***Participants***

In this study participated 268 2$^{nd}$ grade students, 471 3$^{rd}$ grade students, 245 4$^{th}$ grade students, 263 5$^{th}$ grade students and 197 6$^{th}$ grade students, who participated in the Kangaroo Contest in Israel (2014). The range of the students' ages was between 7 and 12 years old and they came from all over the country, from different social backgrounds, from large cities as well as small villages.

### *Tools*

As described above, each student coped with a 75-minute test prepared by the Kangaroo International Committee. Note that 3rd grade students had the same test as 4th grade students and the 5th grade students had the same test as the 6th grade students.

In each test, there were a different number of problems oriented on spatial abilities:

2nd grade - 13 SA problems of 24 problems;

3rd and 4th grade - 5 SA problems of 24 problems;

5th and 6th grade - 5 SA problems of 30 problems.

According to our research, all problems were divided into two groups: (1) SA problems – such tasks that demand spatial abilities and (2) rest non-standard problems (RNSP) – based on the next topics: common sense (logic), number sense and word problems.

Each correct answer granted the students 1 point while no points were received in other case.

### *Data Collection and Analysis*

For each grade, we found the mean and the standard deviation of the SA tasks as well as the mean and the standard deviation of RNSP in the same test. Then the Pearson correlations between the scores of SA tasks and the RNSP in the test were found for each grade.

All collected data is presented in Table 1:

| Grade | Number of students | Number of SA tasks | Mean (St Dev) | | Pearson Correlation between SA tasks and the RNSP |
|---|---|---|---|---|---|
| | | | SA | RNSP | |
| 2 | 268 | 13 of 24 | 0.4446 (0.1788) | 0.4054 (0.1670) | 0.552** |
| 3 | 471 | 5 of 24 | 0.4017 (0.2416) | 0.3730 (0.1680) | 0.436** |
| 4 | 245 | 5 of 24 | 0.5649 (0.2629) | 0.5334 (0.1970) | 0.551** |
| 5 | 263 | 5 of 30 | 0.3510 (0.2230) | 0.3608 (0.1519) | 0.435** |
| 6 | 197 | 5 of 30 | 0.4213 (0.2123) | 0.4319 (0.1541) | 0.441** |

$**p \le .005$

Table 1. Comparing scores in SA tasks vs RNSP in the Kangaroo Contest

### 2nd Grade Data

In 2nd grade 13 of 24 tasks were focused on spatial abilities. The large number of the SA tasks in the test can be explained by the statement that students in this age still have difficulties in reading and understanding of textual problems and their arsenal of mathematics tools is still poor. The means of two groups of tasks (SA and RNSP) for 2nd grade were close: $\bar{x}_{SA}(2^{nd}\ \ grade) = 0.4446,\ \ s = 0.1788$ and $\bar{x}_{RNSP}(2^{nd}\ \ grade) = 0.4054,\ \ s = 0.1670$ . We have found a strong correlation between these scores: $r = 0.552$ and $p \le 0.005$.

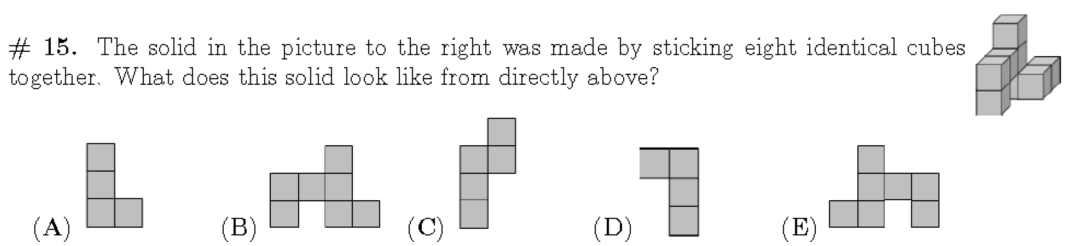
### 3rd and 4th Grades Data

There were five SA tasks in common Kangaroo tests for 3rd grade and 4th grade. The means of two groups of tasks (SA and RNSP) for 3rd grade were close: $\bar{x}_{SA}(3^{rd}\ grade) = 0.4017,\ \ s = 0.2416$ and $\bar{x}_{RNSP}(3^{rd}\ grade) = 0.3730,\ \ s = 0.1680$. We have found strong correlation between these scores: $r = 0.436$ and $p \leq 0.005$.

The means of two groups of tasks (SA and RNSP) for 4th grade were also close: $\bar{x}_{SA}(4^{th}\ grade) = 0.5649,\ \ s = 0.2629$ and $\bar{x}_{RNSP}(4^{th}\ grade) = 0.5334,\ \ s = 0.1970$. We have found strong correlation between these scores: $r = 0.551$ and $p \leq 0.005$.

The scores of the same set of problems in different grades were significantly different. The mean of the set of SA problems for 4th grade was $\bar{x}_{SA}(4^{th}\ grade) = 0.5649$ and it was 40% more than the mean of the same set for 3rd grade: $\bar{x}_{SA}(3^{rd}\ grade) = 0.4017$.

After detail analysis of the differences between the 3rd and 4th grades scores of SA tasks, we discovered that in all five SA tasks, the 4th grade students had better performance than 3rd grade students and the largest gap was found in question #15 (belongs to the Spatial Orientation):




Pic 1. Task # 15 suggested for 3rd and 4th grades students in the Kangaroo Contest 2014

The mean of 3rd grade students of this task (# 15) was: $\bar{x}_{N=471}(3^{rd}\ grade) = 0.1975$ whereas the mean of 4th grade students of the same task was approximately 70% bigger: $\bar{x}_{N=245}(4^{th}\ grade) = 0.3347$. The distribution of chosen distractors per grade was the following:

| Distractor | A | B | C (correct answer) | D | E | No answer |
|---|---|---|---|---|---|---|
| 3rd grade (N=471) | 2.3% | 53.9% | 19.7% | 1.5% | 15.1% | 7.4% |
| 4th grade (N=245) | 1.6% | 47.8% | 33.5% | 1.6% | 12.3% | 2.4% |

Table 2. The distribution of answers in Task #15 for 3rd and 4th grades

We can see that in both grades the distractor B pulled the attention of about half of each of two populations of students. Some explanation for this may be that students probably looked at the shape from the right side while the question addressed to the view above.

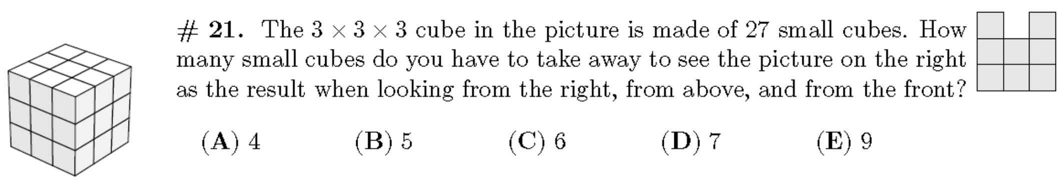
## 5th and 6th Grades Data

There were five SA tasks in common Kangaroo test for 5th grade and 6th grade. The means of two groups of tasks (SA and RNSP) for 5th grade were close: $\overline{x}_{SA}(5^{th}\ \ grade) = 0.3510,\ \ s = 0.2230$ and $\overline{x}_{RNSP}(5^{th}\ \ grade) = 0.3608,\ \ s = 0.1519$. We have found strong correlation between these scores: $r = 0.435$ and $p \le 0.005$.

The means of two groups of tasks (SA and RNSP) for 6th grade were also close: $\overline{x}_{SA}(6^{th}\ \ grade) = 0.4213,\ \ s = 0.2123$ and $\overline{x}_{RNSP}(6^{th}\ \ grade) = 0.4319,\ \ s = 0.1541$. We have found strong correlation between these scores: $r = 0.441$ and $p \le 0.005$.

The scores of the same set of problems in different grades were significantly different. The mean of the set of SA problems for 6th grade was $\overline{x}_{SA}(6^{th}\ \ grade) = 0.4213$ and it was 20% more than the mean of the same set for 5th grade: $\overline{x}_{SA}(5^{th}\ \ grade) = 0.3460$.

We discovered that in all five SA tasks, the 6th grade students had better results than the 5th grade students and the largest gap was in question #21 (belongs to the Spatial Orientation):



# **21.** The $3 \times 3 \times 3$ cube in the picture is made of 27 small cubes. How many small cubes do you have to take away to see the picture on the right as the result when looking from the right, from above, and from the front?

**(A)** 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 9

Pic 2. Task # 21 suggested to 5th and 6th grades students in the Kangaroo Contest 2014

The mean of 6th grade students of this task (# 21) was: $\overline{x}_{N=197}(6^{th}\ \ grade) = 0.2030$ was approximately 30% bigger than the mean $\overline{x}_{N=263}(5^{th}\ \ grade) = 0.1558$ that received by 5th grade students of the same task. The distribution of chosen distractors per grade was the following:

| Distractor | A | B | C | D (correct answer) | E | No answer |
|---|---|---|---|---|---|---|
| 5th grade (N=263) | 20.5% | 17.9% | 17.1% | 15.6% | 17.9% | 11.0% |
| 6th grade (N=197) | 13.7% | 15.2% | 19.8% | 20.3% | 18.8% | 12.2% |

Table 3. The distribution of answers in Task #21 for 5th and 6th grades

We found that in all grades there was strong correlation between SA tasks and the RNSP in the test. The differences between the means of SA and RNSP in each grade were not significant. We also found a positive relation between the age of a participant and the score in the SA tasks.

## CONCLUSION

In this study, we checked the correlation between solving SA tasks and RNSP in the Kangaroo Contest for grades 2-6. We found strong correlation between scores received by participants in the Kangaroo Contest when coped with SA tasks and RNSP in the same test.
Previous research has established a link between spatial ability and mathematics learning - both categories, children and adults with better spatial abilities, also have higher math scores (Delgado & Prieto, 2004; Lubinski & Benbow, 1992; Robinson et al.,1996).
Many studies indicate that spatial thinking and mathematics are related, especially in early grades, and early intervention is critical for closing achievement gaps in math (Duncan et al., 2007; Jordan et al., 2009; Klibanoff et al., 2006; Starkey et al., 2004).
Cheng & Mix (2014) showed that appropriate development of spatial thinking can improve mathematics learning in children aged 6 to 8 years. The meta-analysis (Uttal et al. 2013a, b) showed that the development of spatial thinking leads to an average improvement of almost 1/2 standard deviation in spatial ability measures, and considering all the aforementioned data, there is excellent basis to hypothesize that spatial training would improve math performance.
The age of the participants in this study was found as a factor influencing success in solving SA problems.
Mix & Cheng (2012) claim the relationship between spatial ability and mathematics performance varies with age. In this study the correlation between scores of SA tasks and RNSP in the contest was strong but not found like depend on participants' age.
An observed correlation between performance on SA tasks and the RNSP tasks in the Kangaroo Contest supports the importance of development spatial ability in mathematical learning of MMS whereas some of them will become later mathematically promising students.

Alternatively to other studies, this study examined (1) the correlation between scores of SA problems and scores of non-standard problems and (2) a population consisting of mathematically-motivated students.
Extending research to samples of different ages, employing longitudinal designs and focusing on gender issues will lead to better understanding of the dynamic nature of mathematic – spatial relationships.
Future research may find it valuable to examine whether some of the components of SA, i.e. *Spatial orientation, Spatial visualization* or *Mental rotation*, influence one or more of the other topics of RNSP: *common sense, number sense* and *word problems*.

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# SHARED CREATIVITY IN MATHEMATICS: THE EMERSION OF COLLECTIVE SOLUTIONS

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***Abstract:*** *We investigated the emersion of creativity in mathematics in relational situations in a group of students of the 5th year of elementary school. Additionally, we suggested a strategy of development of collective creativity in which, through interaction and cooperation, students co-construct solutions to open problems, either through the presentation of solutions to the problems, or through the improvement or judgment of the solutions of others. The emersion of creativity seemed to be favored when the symmetry of power enabled the negotiation of communication.*

*Key words: creativity in mathematics, shared creativity, creativity strategy.*

## INTRODUCTION

It is our purpose to investigate how creativity emerges not from "individual minds" (Glăveanu, 2014), but from the relational processes in which a collective works to find appropriate and original solutions to certain mathematical problems. We assume this perspective considering that the works in the field of mathematics are still recurrently based on studies developed through the application of tests and scales of creativity and that few works have dedicated to the collective dimension of the creative process.

In addition to showing a gap in research in mathematical creativity, the study of collective creativity can be justified by the complexity in which society finds itself, demanding creative people and adapting to teamwork. There is a consensus among researchers (eg Alencar & Fleith, 2003; Van den Bossche et al., 2011) that the diverse environments in the world have been configured and reconfigured at an impressive speed and that "creativity is a personal and social trait which promotes human progress at all levels and at all points in history "(Leikin and Pitta-Pantazi, 2013, p. 159). And in this process, they attach an important role to mathematics insofar as the development of this area of knowledge facilitates technological and scientific progress.

Thus, people are becoming more involved in work activities that demand collective work. Van den Bossche et al. (2011) justify these new job configurations postulating that teams are increasingly being used to discuss and manage complex problems. The present work falls into the midst of these two recent concerns: the development of creativity and teamwork. Therefore, we set the objective for this research:

a) to analyze the process of emersion of creativity in mathematics in a group of students and

b) present a strategy of development of collective creativity in which everyone can contribute in some way, either by presenting solutions to the problems solved or by improving or judging the solutions of their peers in the group.

## Mathematical Creativity

Although they are gradually gaining ground in research fields, studies on creativity in mathematics up to that point are inconclusive, and as for the construct, they point to a plurality of definitions (Pitta-Pantazi et al., 2013, Mann, 2005). In our studies, we consider creativity in mathematics as:

> The ability to present numerous possibilities for appropriate solutions to a problem situation, So that these focus on distinct aspects of the problem and / or differentiated ways of solving it, especially unusual shapes (originality), both in situations that require the resolution and elaboration of problems and in situations that require the classification or organization of objects and / or mathematical elements in function of their properties and attributes, whether textually, numerically, graphically or in the form of a sequence of actions. (GONTIJO, 2006, p. 4)

In the literature of creativity in mathematics, one can perceive the preponderance of two aspects: a) the analysis of creativity measures through tests and b) the consideration of individual aspects "and the little recognition of the phenomenon when collective" (Rover & Carvalho, 2006, p. 7). Thus, literature lacks research that seeks to understand collective creativity, since current studies have not necessarily investigated mathematical creativity as a collective process (Levenson, 2011).

## Shared Creativity

The emersion of creativity, that is, its passage from the individual state, solitary to the occurrence in a team context, in solidarity, arises as a research need that can offer an understanding of how such a phenomenon occurs. We focus on the school, a meeting place for learners who are organized in groups and who can form groups by affinity or by other criteria, and mathematics as a specific area in which an intervention activity is proposed for the emergence of creativity. To do so, we chose to understand creativity from the perspective of shared cognition, a construct that emerged in the context of research in organizational psychology for over 20 years (Cannon-Bowers & Salas, 2001). In this work, we consider shared cognition as "sharing and / or congruence of knowledge structures that may exist at different levels of conceptualization within a group and relate to aspects of the group task" (Swaab et al., 2007, p. 188).

## METHODOLOGY

We use the quantitative analysis of the solutions presented by the students and the participation of each of them in their team through the frequency of participation in the solutions in order to characterize how each respondent participated in the process of creating figures. We also use qualitative procedures to analyze the effectiveness of the collective creativity strategy used. A group of fifth grade students from elementary school and the class teacher from a public school participated in the study. We used items from a Shared Creativity Test in Mathematics (Carvalho, 2016) and questionnaires for group configuration as data collection instruments. The test is composed of open questions and assesses collective creativity in terms of fluency, flexibility and originality containing items on resolution, formulation and redefinition of problems (Haylock, 1997; Gontijo, 2007). The item of the test analyzed here and the questionnaires are presented following. The present data are part of a larger doctoral study and, therefore, we present here only partial results that occurred during the validation process of the Shared Creativity Test. We chose a small number of participants to preliminarily analyze the potentiality of the test as a collective, fluid, flexible and original solution production tool. Therefore, the conclusions presented here relate to the group analyzed and do not intend to serve as parameters for generalizations. Our intention is only to show evidence of how the process of emersion of creativity in mathematics can occur from a few participants.

## Procedures

For this work, a random cut of the totality of participants was selected. Thus, we present the solutions obtained by a group of three students, two girls and one boy, all ten years old. The first stage of the research was aimed at the identification of the students in order to gather information for the configuration of the groups. The class mapping questionnaire was applied in which the students were asked to answer the following question:

**1- Imagine that the teacher will perform a group math activity and ask you to choose two people with whom you want to do this activity. Who would you choose?**

**2- Why did you choose these students?**

It was found that the great majority of the respondents chose three students who considered better in mathematics.

The second stage of the research consisted in the application of a strategy of development of collective creativity based on the collaborative learning model of Van den Bossche et al. (2011), The first of which is that the learning of teams occurs through a social process in which knowledge builds up mutually through cognitive sharing, which can only occur in situations of interaction in which everyone can contribute according to their personal styles. The item was as follows:

**1- Using the geometric figures below, build as many figures as you can by following the guidelines:**
**2- All figures must be used;**
**3- They are not worth abstract figures;**

**4 - The figures can be superposed, rotated and must touch at least one point;**

**5- On a separate sheet, give titles to the pictures that describe as much as possible what you wanted to represent.**

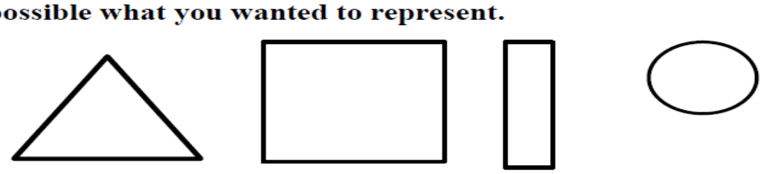


Figure 1 - Shared Math Creativity Task

This strategy occurs in three stages.

**Stage 1.** Free and individual production of figures in ten minutes. After this time, each student names their production by writing a title for the pictures.

**Stage 2.** Students are grouped together observing the figures produced by the other participants and speaking what he thinks each one means.

**Stage 3.** Each author reveals what he wanted to represent with each figure so that his teammates can judge production by accepting it, improving some aspects or modifying the initial idea.

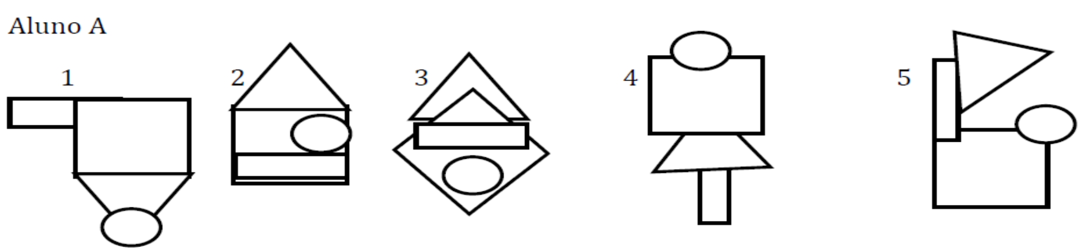
## Results

The following table reproduces the frequency of the solutions presented, the solutions validated in pairs, the modified solutions and the participations in which they judged (emitting opinion in stage 2) and modified the solutions of the partners. In this work, we will present only the frequency of opinions issued by students B and C in relation to the solutions presented by student A. The same will be done in relation to the suggestions of modification offered to student A solutions, however, the student himself, sharing his colleagues' suggestions and rethinking his productions, also began to propose modifications to his initial solutions.

| | Presented Solutions | Validated Solutions | Modified Solutions | Opinion Issued | Suggestion for modification |
|---|---|---|---|---|---|
| Student A | 5 | 5 | 5 | – | 2 |
| Student B | 3 | 2 | 0 | 5 | 4 |
| Student C | 7 | 6 | 2 | 4 | 3 |

Table 1: Frequencies of solutions and participations

To follow, the solutions proposed by student A with the corresponding modifications that have resulted in the final product.

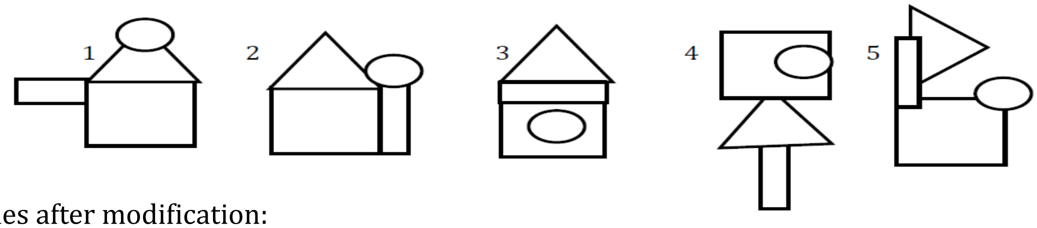



Titles:

1 - War Car, 2 - Toy house, 3 - Clown with glasses and hat, 4 - Tree house, 5 - Boat with a mermaid on the tip

Figure 2 - Student A productions

None of the solutions presented by student A were accepted without some contribution from peers. For example, in Figure 3 the agreement between the members made it possible to modify the clown's hat under the argument that the hat was Chinese and not a clown. The following is the result of the improvement and modifications of student responses A:




Titles after modification:

1 - Pan, 2 - House with pole, 3 - Clown of glasses and hat, 4 - Safe hidden in the tree, 5 - flag of game of golf

Figure 3 - Results of the negotiation stage

Another interesting result can be found in some student A speeches when discussing their productions. While still defending his point of view, the student began to reflect when being evaluated by his peers.

“I agree with you. Thinking well, I think I'd better change my hat.”

“You gave me a great idea. I have not thought about it. Some wheels are missing.”

## Discussion

Analyzing the data, one can see the fact that the asymmetric power relation, initially pointed out by them when answering the questionnaire about the group configurations they conceived, started to take on another meaning: the students who were heavily harassed to participate in the Groups of their colleagues, in this specific activity, began not to monopolize the cognitive processes of the classroom. We realized that there was not a mere reception of ideas of students with greater performance on those with difficulties in this discipline, but an instigating negotiation in order to defend diverse points of view and to construct a set of solutions more cleared and collectively produced. The students began to re-evaluate their own productions after listening to the suggestions of their colleagues, including presenting themselves suggestions for their initial productions (see the speeches descriptions in Results).

It is also noticeable, the importance of the variety of cognitive styles so that a group can produce clear solutions and add qualitative gain. Following the typification of Sternberg's creative styles (1991), there are three types of styles: legislative (preference for formulating problems and creating new solutions and ways of seeing things), Executive (primacy for implementation of ideas created by third parties) and judiciary (a propensity to evaluate others and express opinions about their productions). In our case, we will judge as characteristic of legislative style the presentation of new ideas, executive the improvement of ideas of third parties, and judicial the issuance of opinions.

In fact, observing table 1, we can see that student A obtained 100% of their solutions and student C presented the highest frequency of solutions, and could be supposed to be identified in this case with a creative legislative style. On the other hand, student B, although presenting a low frequency of solutions, was prone to judge and to make modifications in the solutions of student A. This plurality of creative styles allowed a cognitive complementation (Cannon-Bowers and Salas, 2001) in which each member collaborated differently for the final outcome of the solutions, suppressing the gaps left by their peers.

Finally, we point out a fragility in our strategy of collective creativity. In judging the solutions, it seems to us that some collective mistakes can hinder good ideas because of the lack of experience with this form of activity. For example, the modification in Figure 5 in which originally student A had considered "boat with a mermaid on the tip" to "golf game flag" to us represents a loss in terms of creativity, since the initial idea seems much more original and interesting than its subsequent modification.

## Conclusion

The emersion of creativity from a solitary and individualistic act to a solidary and interactive process creativity seemed to be favored, in this case, when the symmetry of power enabled negotiation communication with equal decisional force, recognition of the possible and necessary changes in the suggestions of others and understanding of which collective production is possible through the collaboration of people with diverse creative styles.

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# ANALOGICAL TRANSFER: HOW FAR FROM CREATIVITY?

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***Abstract.*** *Analogical thinking is a natural form of transfer. The paper reports on mathematically talented students' behavior in a problem-posing context focused on analogical transfer. Findings indicate that, in a sample of high achievers, the use of analogy in posing a new problem related to a given one allows characterizing students' cognitive flexibility, and thus differentiating more mathematically creative persons.*

*Key words:* Problem posing, analogical transfer, creativity, cognitive flexibility.

## INTRODUCTION

Research shows that transfer is hard to come by, particularly far transfer (e.g., Salomon & Perkins, 1989; Perkins & Grotzer, 1997; Singer, 2008). In a study made on experts, Dunbar (2001) has stressed that analogy (as a particular form of transfer) is easy in naturalistic settings, yet quite difficult in the laboratory. A large gamut of research was concentrated to uncover aspects of the process of analogical problem solving. The different sub-processes of analogy have been explored empirically, among which: representation (e.g., Novick, 1988), analogical mapping (e.g., Clement & Gentner, 1991), adaptation and induction (e.g., Novick & Holyoak, 1991).

Analogical thinking is natural, people spontaneously make analogies. Starting from this premise, we were interested to study how students use analogy in the context of mathematical problem posing and if the realization of analogies can be correlated with creativity. Specifically, the starting questions of this study are: What are the processes through which analogical transfer happens (if the case) in a problem posing context? In what way this type of transfer relate to creativity?

### Analogical transfer

According to a large body of literature (e.g. Copi & Cohen, 2005), an analogical argument has the following shape:

1 *S* is similar to *T* in certain (known) respects.
2 *S* has some further feature *Q*.
3 Therefore, *T* also has the feature *Q*, or some feature *Q** similar to *Q*.

(1) and (2) are premises, and (3) is the conclusion of the argument. The argument form is *inductive*; the conclusion is not guaranteed to follow from the premises. *S* and *T* are referred to as the *source* (or *base*) *domain* and *target domain*, respectively. A *domain* is a set of objects, properties, or relations, together with a set of accepted statements about those. More formally, a domain consists of a set of objects and an interpreted set of statements about them. Formally, an analogy between *S* and *T* is a one-to-one mapping between objects, properties, and relations in *S* and those in *T*. Not all of the items in *S* and *T* need to be placed in correspondence. Commonly, the analogy only identifies correspondences between a selected set of items (Bartha, 2013).

## Creativity and problem posing

Mathematical creativity received much attention from researchers who focused on defining or characterizing it, or on establishing criteria for its assessment (see, for example, Sriraman, 2004). In the present paper, we use a framework based on the concept of cognitive flexibility to study creativity in a problem posing context (Pelczer, Singer, & Voica, 2013; Singer & Voica, 2015; Voica & Singer, 2013). We consider that a student proves cognitive flexibility when (s)he generates new proposals that are far from the starting item (i.e. cognitive novelty), poses different new problems starting from a given input (i.e. cognitive variety), and is able to change his/her mental frame in solving problems or identifying/discovering new ones (i.e. change in cognitive framing, or reframing).

# METHODOLOGY

During a summer camp with students in grades 4-12 (9-18 years old), we initiated a call for problems. We asked participants to choose a problem from a list of three multiple-choice problems (accessible for all the students in the camp as level of difficulty and complexity of knowledge) and, after they solve it, they have to pose a new problem, obtained by modifying the chosen one. Students had three days to formulate their proposals. In total, 176 students submitted problems.

The instructions given to students stipulated that changes on the initial problem can be made through a generalization, a completion, an application, or even a more particular case. The focus of this task was less related to how students pose problems, but to the analysis of students' cognitive behavior in a context of transfer. The new posed problem was supposed to be formulated in the same way as the chosen one (i.e. multiple-choice problem, with 5 possible answers, and a unique correct answer) and be accompanied by a solution.

The students in the camp were selected based on a two-round national competition – first round at a local level, and a second round – very selective. In the camp, the winners of this last contest were invited (1% of all participating students). Therefore, we consider that students from the camp were promising mathematically talents.

In this paper, we analyze the proposals received from students who have chosen the following problem (hereinafter called the starting problem); this was the Problem no. 1 in our call-for-problems list:

> Consider the following associations: $36 \to 18$; $325 \to 30$; $45 \to 20$; $30 \to 0$.
>
> Find a rule that describes these associations and specify, within this rule, the number that matches 531.
>
> a) 20; b) 40; c) 15; d) 31; e) 51.

In total, for this problem, we got 99 posed problems from 97 students (two students posed 2 problems). These students represent our sample.

The starting problem is open-ended: starting from the given associations, one can imagine various association rules. To keep the problem to an affordable level of difficulty, we chose input numbers and distractors so that to suggest solvers the following rule (considered by us as being the most accessible to students): a natural number matches the product of its digits. In this way, we have obtained from our sample of students a similar behavior: all respondents identified this rule in the starting problem. Our interest in this task comes from the fact that it captures analogical transfer: solving this task involves exploring the initial context (i.e. understanding the given examples, proposing a pattern, checking its functionality on new cases), followed by managing the same stages within another context, imagined by the solver. In other words, students had to move from a source domain (i.e. the starting problem and its context), to a target domain (i.e. the new posed problem and its context), making an analogical transfer between domains.

Each of the posed problems was initially evaluated using the following criteria: originality of the proposed pattern, accuracy formulation, and correctness of the solution. After the first raw-data analysis, we further classified the posed problems into clusters using two criteria (with several sub-criteria): nature of the correspondence rule, and multiplicity of the correspondence rule. The nature of the correspondence rule refers to the type of correspondence envisioned by a student. From this perspective, correspondences we found within the sample were generated by an algebraic operation, or by a function, or they were representational, or alphanumeric. The multiplicity of the correspondence rule refers to the universality of the one-to-one association: a student may propose a single rule that applies to all the input elements, or multiple rules that require partitioning the data domain into at least two subsets. The analysis that we did is a qualitative one, based on students' written submissions.

## RESULTS

We present in this section some relevant examples.

*Example 1: Correspondence of algebraic-operation type*. Cristian (grade 9) posed the following problem:

> I consider the associations: $30 \rightarrow 3$; $25 \rightarrow 1$; $36 \rightarrow 2$; $7 \rightarrow 1$; $210 \rightarrow 4$.
> Find the rule and state the number that matches 16.
> a. 2; b. 5; c. 3; d. 4; e. 1.

The rule specified by Cristian associates to a natural number the number of prime factors of its prime factorization. Cristian has moved further from the given pattern; he kept in the background the product idea, but he used prime factorization, not the number digits. We classified this problem in the category "correspondence of the type algebraic-operation" because the proposed rule is based on decomposing the numbers of the input set (i.e. prime factorization) and this is utilized to perform algebraic operations with the identified compounds.

*Example 2: Functional correspondence*. While some students imagined decomposing rules, other students processed input data as a whole. We included correspondences of this kind in the category of functional correspondences. An example of this category is the problem posed by Emoke (grade 6):

> Consider associations: $36 \rightarrow 24$; $45 \rightarrow 30$; $30 \rightarrow 20$; $312 \rightarrow 208$; $123 \rightarrow 82$.
> Find a rule that describes these associations and specify the number that matches 531.
> a. 77; b. 177; c. 154; d. 354; e. 508.

Emoke uses the rule: the natural number $n$ matches the number $2n/3$. She kept the question of the starting problem (including the number 531), modifying distractors accordingly. One of the distractors (177) corresponds to the result of dividing 531 to 3. In her posed problem, Emoke looks at the number input as a whole and applies a rule of association in which various components of the input numbers (digits, groups of digits, factors) does not play any role.

*Example 3: Representational correspondence*. Ruth (grade 4) posed the following problem:

> The first 60 terms of a sequence are: 000000; 000001; ...; 000 059.
> What is the next number in the sequence?
> a) 000001; b) 000100; c) 010000; d) 010 059; e) 000101.

Ruth placed the terms of her sequence in relation to a clock display (*hh-mm-ss*). She moved with her posed problem in a new context, in which mathematical symbols changed their usual meaning. We classified this problem in the category "representational correspondence", because the proposed rule cannot be understood in the absence of the representation of numbers on a clock display.

*Example 4: Correspondence of alphanumeric type*. Cosmin (grade 5) posed the following problem:

> The following numbers are associated: $4 \rightarrow 5$; $7 \rightarrow 5$; $9 \rightarrow 4$; $11 \rightarrow 10$.
> Find the rule and the number associated with 15.
> a. 8; b. 14; c. 9; d. 13; e. 6.

Cosmin stated that the problem may seem difficult, but it is actually about letters; more specifically, a natural number is matched with the number of letters of the linguistic utterance of that number in the Romanian language. (For example, 4 = "patru" has five letters, therefore $4 \rightarrow 5$.) Cosmin posed "alphanumeric type correspondences" in which numerical values were assigned to the letters based on some underlying rules.

## DISCUSSION AND CONCLUSIONS

The students of our sample posed new problems based on a given problem. For doing so, they had to understand the source domain - represented by the context of the starting problem and to put into action a mechanism for the replication of the initial problem, focusing a target domain - represented by the new proposal. All these show that students in our sample managed to do analogical reasoning. In the specific case of this study, the final construct they realized (the new problem) involves itself analogical transfer, meaning that students must identify a context in which they apply an iterative rule of association, which has to satisfy all terms of a sequence, therefore some generalization steps are embedded into each consistent proposal. The fact that students are able to pose new problems, in which they imagine various rules of correspondence, shows that analogical transfer functions, with different degrees of getting far from the model.

People spontaneously make analogies. Starting from this premise, we should see to what extent students are creative, in other words to what extent they manage to "transform analogy into metaphors", where the target domain is less explicit, more distant, more structurally variable and can even violate structural consistency (Gentner, Bowdle, Wolff, & Boronat, 2001). Consequently, in the given mathematical context (of problem posing), we analyze students' proposals in relation to the distance between the given model frame and the new frame proposed by each student. In the situation described in this paper, in which various constraints appear due to the task, a creative behavior requires the student to propose surprising rules and to change substantially the sets on which they apply those rules. We further analyze students' creativity from the perspective of cognitive flexibility (i.e. cognitive variety, cognitive novelty, and change in cognitive framing. The children in our study were able to build analogies, which means that they possess the cognitive frame needed for understanding and processing the starting problem. Data analysis (of the students' posed problems) shows that some of the students were able to make major changes of the initial frame, while some did not, staying close to the given frame.

In some cases, the three components of the starting problem (related to the input set, the output set, and law of association) were minimally modified, as in the case of Cristian. In some cases, all these components were kept, but the problem question was changed; in some other cases, the law of association was replaced (e.g., instead of the product of the digits numbers, the sum of the digits of input numbers was used). The problems thus obtained can be difficult, even if the transfer is minimal.

In other situations (as in the cases of Ruth or Cosmin), the changes are more extensive. Cosmin made a far transfer, introducing elements from another domain (linguistics). In addition, he consciously and purposely changed the initial frame, seeking to build a meaningful problem beyond mathematics. Therefore, cognitive reframing could be identified in these cases, acting for enlarging the distance between the base and target domains.

In some cases of far transfer, while keeping the analogy, students applied further restrictions to their proposals, such as, for example, avoiding results that exceed the set of natural numbers, although some may have corresponding output data in Z or Q. This restriction is surprising when taking into account that among the students in the sample there were high-school graders, meaning that they usually operate with real numbers. To fulfill the condition of keeping the output results in N, some students restrict the input set of the posed problem. Thus, we can see how Emoke has chosen to apply the functional relationship $n \rightarrow 2n/3$ only to integers divisible by 3. This students' caution, manifested within the entire group, seems to act as a limitation on their ability of transfer, and as a constraint on cognitive novelty, as well. However, at the same time, it also indicates the students' tendency to ensure that the analogy is valid through a bi-directional check source-target.

What about cognitive variety? We note that the starting problem involves a complex of responses (it is a system with correlated elements, to be connected through a pattern). In this case, there was no point to ask students to propose as many as possible correspondences - as it is not relevant: once a student has proposed a surprising rule, with possibilities of generalization that operates on a whole class of elements, he/she proves potential cognitive variety. On the other hand, in the responses we received we found a wide variety of association rules: practically within the whole sample, only two students have proposed the same association rule. We interpret this dispersion of students' proposed patterns as an argument that the sample denotes cognitive variety, as a group.

Concluding, our study shows that, in a population of high achievers, the use of analogy allows differentiating more mathematically creative persons. Motivation appears to be an important factor, which put some students to make far associations and to strive giving meaning beyond the mathematics involved in their proposal.

This qualitative study is more focused on the cognitive side of the potential and the limits of analogy for reasoning, rather than on distinguishing students' personal features related to creativity. Our study shows that it is important to exploit the natural ability of human thought to use analogies. Analogical thinking should be used more often and explicitly; a way to use it is engaging students in building mathematical patterns. By acknowledging possible relationships between a source and a target domain, given explicitly or only implicitly, students learn to build new situations/contexts/problems and to control distances and relationships among the terms they use, thus engaging and training their mathematical creativity. This extrapolation requires, however, new studies and focused research questions.

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# WORKSHOPS

# DESIGN OF PROBLEMS FOR RESEARCH PURPOSES WITH MATHEMATICALLY TALENTED STUDENTS

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***Abstract.*** *In this workshop, through specific examples, we propose the attendees to "create" their own problems so that they fit specific research needs. The process for the design is based on reviewing the appropriate bibliography, consulting databases and programs addressed to talent, selecting related problems and modifying their components. Participants in this workshop will be organized in small groups and will make their own proposals, which will be discussed in the whole group, where the proposed schema for the design of tasks related to specific research will be completed.*

*Keywords: mathematical talent, problem solving, visualization, cooperative work, resolution strategies, problem design*

## INTRODUCTION

Problem solving acquires a relevant role in the research with mathematically talented students, and it is used both in identification, characterization, and intervention (Davis, Rimm & Siegle, 2014; Neider & Irwin, 2001). A close correspondence between research objectives and characteristics of the problems used as instruments is established. The researcher has access to a bank of resources from bibliographic reviews, specific programs addressed to mathematically talented students, mathematical competitions, etc. But sometimes, the result of this search does not suit the specific needs of the study, and then the researchers are led to "create" their own problems.

A strategy to design problems that best fits the objectives of a specific research is to modify some elements of other problems. It is necessary a detailed analysis of the variation which results from each modification; this enriches the knowledge of the researcher as an expert in the resolution of this specific problem.

## PROBLEM DESIGN

There is an initial scheme in which various phases of work are sequenced in order to create the problem that best fits the proposed research.

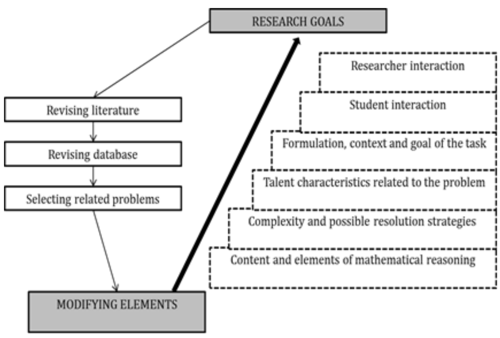


Figure 1. First diagram to design problems

In this workshop, we propose as an example some of the specific objectives of a piece of research that we are carrying out at present. It concerns rich tasks for mathematically talented students that allow the analysis of the skills of visualization and visual argumentation, to analyze the transmission of information to solve a cooperative task and to analyze the strategies used in proof (Beltran-Meneu, Ramirez, Gutiérrez & Jaime, 2016, Ramírez, 2012).

In relation to the review of the literature, for each research aim we will present the participants problems considered in similar research, as well as a search in the databases of problems in the NRICH project (University of Cambridge http://nrich.maths.org) and in the International Mathematical Olympiad (https://www.imo-official.org).

For the modification of the problems, we will take into account the following elements: formulation, context and goal of the task; content and elements of mathematical reasoning; complexity and possible resolution strategies; talent characteristics related to the problem; student interaction among students and researcher interaction.

In this workshop we will offer the participants some problems to modify the elements and progressively adjust them to the proposed objectives. Subsequently, after working in small groups and a general discussion, the proposed variations will be analyzed and an evaluation of the richness of the problem will be done regarding the research intention. Finally, the scheme presented in Figure 1 should be enriched with other possible strategies addressed to create problems or with some variations of the elements that characterize them.

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# PROBLEM SOLVING IN CONTEXT: A VARIETY OF EXTENSIONS OF PROBLEMS

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***Abstract.*** *The idea of the workshop consists in presenting several lessons for children aged 16-18. Each lesson will start with a mathematical problem which further will be generalized in different senses. Most of these problems are taken from the book by Ivanov (2017). It is assumed that the participants will take an active part in solving the problems and discussing the related pedagogical ideas. Moreover, we hope that sometimes the participants will determine the development of the suggested theme.*

*Key words: teaching and learning methods, problem solving, creativity, generalizations.*

The opportunity for effective teaching, especially of mathematically gifted students, depends on mathematical richness of proposed tasks. It is the mathematical content that lays a basis for teaching situations promoting students' creativity, in particular, fostering their ability to see the underlying idea and to catch the given "hints". We are going to outline several lessons starting with rather simple mathematical problems which further will, so to say, be "extended". For example, the natural solution of Problem 1 reveals a deep general idea by which many other problems could be solved. On the other hand, Problems 9 and 10 lead to interrelations between various mathematical notions. Altogether, the authors' approach promotes the development of "flexible thinking", the characteristic of thinking important for a creative work (Krutetskij, 1976).

The theme of Lesson 1 is "Surprisingly short solutions of the geometric problems". We start with the following problem.

Problem 1. Consider the circles with centers at the points $O_1(-1,1)$ and $O_2(3,2)$ with respective radii $r_1 = 3$ and $r_2 = 2$. Find the equation of the straight line through the points of intersection of these circles.

Please, find the solution of this problem "in one line". Do you see the connection of this problem with the following problems?

Problem 2. Suppose we are given three pairwise intersecting circles in the plane. For each pair of circles, consider the line through the two points of intersection of those circles. Prove that if no two of these three lines are parallel, then they are concurrent.

Problem 3. Suppose that each arm of an angle intersects with both arms of another angle. Prove that if bisectors of the two legs are perpendicular, then the four points of intersection of the respective arms lie on a circle.

In mathematics it often happens that making what seems a small change in the formulation of the problem results in a considerable increase in difficulty.

The title of Lesson 2 is "Let us reformulate the problem, or: How to solve the unsolvable equations?"

Problem 4. Find all right triangles with sides of integer length for which the hypotenuse is one unit longer than one of the legs.

If we reformulate this problem in a usual way, we'll arrive at Pell's equations.

Next problem is well-known and has a surprising answer.

Problem 5. A rope is tied around the Earth's equator and then lengthened by 6 feet. How high can the rope be raised off the equator to the same height all the way round? In particular, could a mouse creep under it?

A natural reformulation of this problems leads to equations which couldn't be solved exactly. Thus, to find the solution one has to use approximations based on Taylor's theorem.

Lesson 3 – "Equations in which "the unknown" is a function". In order to lead students into a discussion of exponential functions it would be appropriate to ask them "What is the basic property of an exponential functions?" Unfortunately, we never get a straight answer to this question. We are going to ask participants "Why the identity $a^{x+y} = a^x \cdot a^y$ is the fundamental property of such functions?"

Here is another problem of that kind.

Problem 6. Find all real-valued functions satisfying both the identities $f(xy) = f(x)f(y)$ and $f(x + y) = f(x) + f(y)$ for all real $x$ and $y$.

Lesson 4 – "Five problems and a function". Do you understand that the following problems are closely linked by a function and a routine calculus method?

Problem 7. Find the largest of the numbers $\sqrt[n]{n}$, where $n = 1,2, \ldots$.

Problem 8. Find all pairs of distinct natural numbers $x$ and $y$ such that $x^y = y^x$.

Lesson 5 – "The variety of methods, or: How many different solutions of a routine problem could you invent?"

Solving one and the same problem in several different ways can be very instructive. By means of the single example one can engage students with a variety of the topics and ideas and demonstrate different approaches to problem-solving.

Problem 9. Determine the number of solutions of the equation $\sqrt{x} + \sqrt{6 - 2x} = a$ as a function of the parameter $a$.

Problem 10. Find the least value taken by the expression $\sqrt{a^2 + 1}\sqrt{b^2 + 9} + \sqrt{c^2 + 25}$ under the condition that $a + b + c = 12$.

We hope that the presented ideas will be also useful for prospective teachers of mathematics and used in some university undergraduate courses in algebra and calculus.

## References

Ivanov, O. A. (2017). *Portal Through Mathematics: Journey to Advanced Thinking.* (Anneli Lax New Mathematical Library). Washington: The Mathematical Association of America, MAA Press.

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# WHEN THE GAME ENDS, THE MATH BEGINS

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***Abstract.*** *Teachers frequently use games to engage students and give them practice in reinforcing mathematical concepts and computation. Too often, however, students do not delve into the mathematics behind the games. In this workshop, participants will begin by playing simple elementary mathematics games, and then use questioning techniques and an open problem-solving/problem-posing heuristic to delve into the mathematics and extend students' reasoning and mathematical creativity.*

*Key words: mathematical games, mathematical creativity, depth and complexity, strategies*

## RATIONALE AND PURPOSE

After over twenty-five years without a formal position on gifted students, in 2016, the National Council of Teachers of Mathematics passed a position statement on *Providing Opportunities for Students with Exceptional Promise* where they noted, "Students with exceptional mathematical promise must be engaged in enriching learning opportunities during and outside the school day to allow them to pursue their interests and develop their talent and maintain their passion for mathematics". Mathematical games are the perfect way to do this, but frequently teachers use games to entertain students and perhaps give them practice to reinforce some computational skills, but all too often they stop there, and then turn to the textbook for the "real" math. The purpose of this session is to demonstrate how games might be used as a springboard for in-depth, creative investigations related to the core mathematical content that can be differentiated to accommodate students with a wide range of mathematical talents, including the most gifted.

## DESCRIPTION OF THE WORKSHOP

The workshop will begin with a card game that has been used with primary students to reinforce simple place value concepts and then extended to journaling. Samples of student writing will be shared. This game will be the introduction to games involving addition and subtraction of two-digit whole numbers, including a demonstration of an online version. Those games will be the catalyst for mathematical problem solving and problem posing using an open-ended heuristic and suggested questions for teachers and students designed to tailor problems to students interests and skill levels. Following this, games will be extended to a range of numbers, including decimals, integers, and positive and negative rational numbers. In addition to a variety of card games, simple well-known games such as tic-tac-toe will also be used to spur in-depth problem solving and problem posing.

Several of the games, problem solving/problem posing strategies and heuristics, and samples of student work were developed as part of the award-winning *Project M$^2$: Mentoring Young Mathematicians* for students in kindergarten through second grade funded by the US National Science Foundation and *Project M$^3$: Mentoring Mathematical Minds* for advanced students in grades three through six funded by the US Department of Education as well as *Math Innovations* for middle grades students.

## PARTICIPANT INVOLVEMENT

Participants will be actively engaged in playing games followed by solving and posing a variety of related mathematical problems using an open-ended problem-solving/problem-posing heuristic. Participants will learn innovative questioning strategies based on a series of "who, what, when, where, why, and how" questions that encourage students to delve deeply into engaging, creative and complex problems and persevere in their solutions using a variety of innovative strategies and models, forming generalizations that lead to proofs and rules, and posing additional questions.

## References

National Council of Teachers of Mathematics. (2016). *Providing Opportunities for Students with Exceptional Mathematical Promise: A Position of the National Council of Teachers of Mathematics.* Reston, VA: NCTM.

*Project M²: Mentoring Young Mathematicians:* *www.projectm2.org*

*Project M³: Mentoring Mathematical Minds:* *www.projectm3.org*

*Math Innovations: Moving Math Forward through Inquiry and Exploration*: www.kendallhunt.com

# USING PRODUCTIVE STRUGGLE TO DECREASE FRUSTRATION AND INCREASE MOTIVATION

Dr. William Renwick Speer
University of Nevada Las Vegas

***Abstract.*** *The National Council of Teacher's of Mathematics 2014 landmark publication, Principles to action: Ensuring mathematical success for all, connects research with practices that are essential for every student with core principles to build a successful mathematics program at all levels. An important aspect of successful mathematics teaching practices proposed in this book is the examination of unproductive and productive beliefs, obstacles, and key actions that must be understood, acknowledged, and addressed. We can't make our students into seekers if we aren't seekers ourselves. In this research-based, practice-oriented presentation we explore the benefits of creating productive struggle with desirable difficulties to help students shake up naïve or loose thinking and to construct "new" knowledge by transfer of related knowledge to new situations.*

*Key words: Modelling; Problem solving; Productive struggle; Motivation; Discourse; Reasoning*

## PRODUCTIVE STRUGGLE – DEFINING CHARACTERISTICS AND BENEFITS (AND RISKS)

Mathematical modeling involves taking a real world situation, analyzing it, and then developing a mathematical model or formula that accurately captures relationships and predicts outcomes. One can explain the basics of mathematical modeling by working through an example using a six-part process. These six components are based on the process that students go through as they attempt to develop a mathematical model.

Purposes for exploring instructional strategies that emphasize productive struggle include:

1. Enabling all students with motivating opportunities to engage in reasoning and discourse about challenging quantitative ideas.
2. Encouraging a better understanding of how concepts and ideas interconnect and build on one another to produce a coherent mathematics knowledge base that effectively integrates concepts of date, algebra, geometry, measure and probability.
3. Recognizing and applying quantitative thinking and deductive reasoning to solve problems, including within contexts that are outside of the "apparent" or "evident" mathematics represented.

The following illustration represents but one example of several problems that will be actively explored with participants through discourse and hands-on engagement at the concrete-manipulative, representative-pictorial, and abstract-symbolic levels.

**Problem: How long would it take to evacuate a classroom in an emergency?**

**a. Attempt to better understand the problem by putting it in your own words.**

How long would it take a people to exit a room for a fire drill?

**b. Considerations and assumptions you might want to know to solve this problem.**

How many people are in the room?; Are the people seated or standing?; Are there obstacles in the way?; How many exits and where are they located?; How fast and orderly do people walk?; Are there disabled people in the room?; What is a person's reaction time?; Have the people had any practice with fire drills?; What other factors might yield a "fastest escape?"

**c. Describe the variables you need and assign letters to each one.**

Number of people in the room (N); Total time to evacuate (T); Distance of first person to exit (D); Average walking speed (S); Reaction time for first person to reach the door (R); Time needed for each person to walk through the doorway (Et)

**d. Solve the problem and create a mathematical model.**

This seems like a very complicated problem at first glance. Each person has a different distance to walk so how can one formula describe all that variance? Consider the following solution: Decide how long it would take the first evacuee to reach the door. After that the rest of the group would be queued up right behind that person ready to exit. It becomes irrelevant how far each person has to walk and the solution can be represented by:

Evac. time = (time to door of evac. #1)+(# of exiters)(time through the door)

**Et = R + (N) x (T)**

What seemed like a very complex problem can be approximated by a linear model. where the y-intercept, slope, and dependent and independent variables all have clear, concrete meanings. The y-intercept is the time it takes the first evacuee to reach the door. The slope is the time it takes each person to walk through the doorway. The independent variable is the number of people in the room and the dependent variable is the time needed to evacuate the room. There are lots of handles for students to grab onto and understand.

There are certainly other models that could be fitted to this situation, which is one of the attractions of this process. Being creative and developing a reasonable model is far more important than getting a particular answer (model).

**e. Test and adjust the model.**

Plug in your numbers and predict how long the evacuation would take. Then evacuate the room and compare your prediction and results. Evaluate the accuracy of your model and accept it, adjust it, or scrap it depending on its goodness of fit.

**Additional SAMPLE Problems**

*In these problems, scenarios are vaguely stated. From these, discussion will serve to identify problems for deeper study. What variables affect the potential outcomes of the problem? Which of these variables are the most important (and why)? How can the problem be "tackled" so that a defensible solution is the result?*

a. A retail store plans to create a new parking lot. How should it be illuminated?
b. How might a manufacturer of a product decide how many items of that product should be manufactured each year and how much to charge for each item?
c. Is college a financially sound investment? What factors determine the total cost of a college education? How would you determine the circumstances necessary for the investment to be profitable?

**Other RELATED "larger-scale" Projects**

a. Consider the taste of brewed hot tea. What are some of the variables affecting taste? Which variables might be initially neglected? Suppose you hold all variables fixed except water temperature. Most teapots use boiled water in some manner to extract the flavour from the tea leaves. Do you think boiled water is optimal for producing the best flavour? How would you test this? What data would you collect and how?
b. A city notices that it is spending too much money on its garbage collection methods. Identify the multitude of variables that might possibly be controlled to save money. What are the advantages and disadvantages to controlling these? What new problems are introduced to the community by attempts to control variables?

## References

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: NCTM.

# DISCOVERIES WITH DGE: WHICH ONE IS MORE CREATIVE?

Roza Leikin & Haim Elgrably

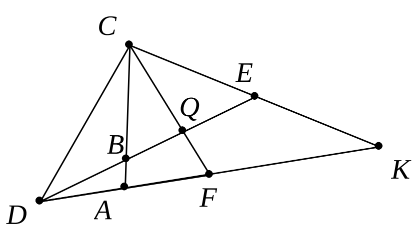
***Abstract.*** *During the workshop we will share with the participants our experience of "teaching to be creative" implemented with prospective high school teachers. We will present different discoveries that our teachers did and ask the participants of the workshop to decide "What is more creative?" We will introduce our scoring scheme for the evaluation of creativity in the discovery activity. The participants will be asked to discover and evaluate their own discoveries.*

In our work an INVESTIGATION TASK is a complex task that includes (Leikin, 2015):

(a) Solving a proof problem in several ways;
(b) Transforming the proof problem into an investigation problem;
(c) Investigating the geometry object (from the proof problem) in a DGE for additional properties (experimenting and conjecturing);
(d) Proving or refuting conjectures.

The following is an example of such a tasks

**Problem 1**




Given:

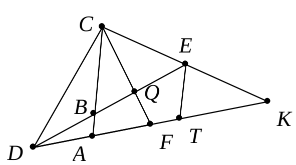
$\Delta$ DCK,
DE median in $\Delta$ DCK
CF median in $\Delta$ DCK
CA median in $\Delta$ DCF

Prove that

$$\frac{DE}{DB} = \frac{5}{2}$$

in at least 2 different ways

**Proof 1.1:** Auxilary construction $ET||CA$




**Proof 1.2:** Auxilary construction $EG$ through $F$ ($G$ on $CA$)

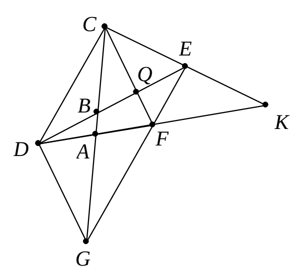
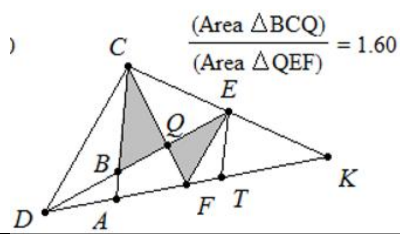
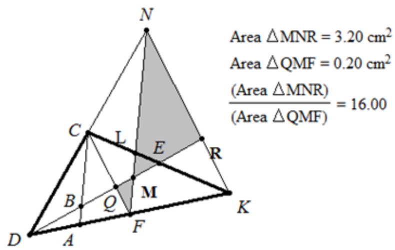
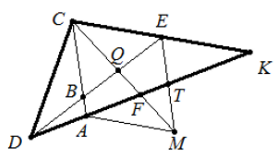



Figure 1: Two ways to proof Problem 1

The following are the examples of the discoveries:

| | | | |
|---|---|---|---|
|   | a. $\frac{BC}{AB} = 4$<br>b $\frac{CQ}{QF} = 2$ | c $\frac{A(BCQ)}{A(EQF)} = \frac{8}{5}$<br>d $\frac{A(DCQ)}{A(EQF)} = 4$<br>g. $\frac{DA}{DT} = \frac{2}{5}$ | e $\frac{AT}{DK} = \frac{3}{8}$<br>f $\frac{QE}{BQ} = \frac{5}{4}$<br>h $\frac{A(DCK)}{A(ETK)} = \frac{16}{3}$ |
|   | j. $\frac{NR}{RK} = 2$<br>k. $\frac{A(MNR)}{A(BCQ)} = 4$ | l. $\frac{A(MNR)}{A(QMF)} = 16$<br>m. $\frac{A(MLE)}{A(QMF)} = 1$ | n. $\frac{A(MNR)}{A(CFK)} = \frac{32}{30}$<br>o. $\frac{A(MNR)}{A(CLF)} = \frac{16}{5}$ |
|   | $EM$ through T,<br>$M = ET \cap CQ$ | *Then*<br>$CAME$ is a parallelogram *and*<br>$\frac{A(DCK)}{A(ACEM)} = \frac{4}{3}$ | |

We ask participants to think which of these discovers are more or less creative.

The participants will search for the new properties of a given figure (with a different mathematical problem). We will present with scoring scheme and aske the participants to evaluate each other discoveries.

We also will discuss different discovery strategies exhibited by experts and non-experts problem-solvers (Leikin and Elgrabli, 2015).

**References:**

Leikin, R. (2015). Problem posing for and through Investigations in a Dynamic Geometry Environment. In F. M. Singer, N. Ellerton & J. Cai (Eds.) *Problem Posing: From Research to Effective Practice* (pp. 373-391). Dordrecht, the Netherlands: Springer.

Leikin, R. and Elgrabli, H. (2015). Creativity and expertise – The chicken or the egg? Discovering properties of geometry figures in DGE. In *the Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education - CERME-9*. (pp. 1024-1031)

# WORKSHOP
# ERASMUS+ FUNDING, IDEAS AND NETWORKING

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***Abstract.*** *Best Ideas and Innovation cannot be developed without cooperation and co-development. This approach is embedded in EU Funding Programmes as it arises from the requirements of consortia development and project management. This Workshop will present the preparation and management of proposals and projects under the ERASMUS+ Funding Programme and will invite participants in developing ideas for new proposals and form on the spot consortia and leaders for the preparation of proposals to be submitted for the 2018 Call for Proposals, always supporting priorities relating to Mathematics and its Education aspects.*

*Key words: Mathematics, Education, EU Programmes, Funding, Proposals, Projects, Networking.*

## DESCRIPTION OF THE WORKSHOP

The workshop content will be based on the ERASMUS+ 2017 Call for proposals and will provide an overview of the following topics

- Introduction to ERASMUS+ Programme
- Objectives / Priorities
- Structure of a proposal
- Eligible activities and expenses
- Tools for background preparation
  -Work distribution to partners
  -Staff cost estimation
  -Operational Expenses
- Outputs and planned tasks
- Budget
- Examples of approved proposals KA2
- Management of the project after approval
- Ideas, Networking and partnership development during the workshop

Focus will be given on the ERASMUS+ Key Action KA2 Strategic Partnerships and KA2 Capacity Building. As these two actions provide for centralized and decentralized proposal approaches and allows for participants from Programme and Partner Countries, it is expected that it could serve the purpose of the workshop for networking.

About half of the time of the Workshop will be devoted in discussing ideas for new projects with monitoring and support by the workshop leader in such a way that ideas and linked to Programme priorities and objectives, are developed to provide innovation and the expected outputs and products can be supported by eligible funding.

## References

European Commission: https://ec.europa.eu/programmes/erasmus-plus/calls-for-proposals-tenders/2016-eac-a03_en

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